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KRIPKE COMPLETENESS OF STRICTLY POSITIVE MODAL LOGICS OVER MEET-SEMILATTICES WITH OPERATORS

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Abstract. Our concern is the completeness problem for spi-logics, that is, sets of implications between strictly positive formulas built from propositional variables, conjunction and modal diamond operators. Originated in logic, algebra and computer science, spi-logics have two natural semantics: meet-semilattices with monotone operators providing Birkhoff-style calculi, and first-order relational structures (aka Kripke frames) often used as the intended structures in applications. Here we lay foundations for a completeness theory that aims to answer the question whether the two semantics define the same consequence relations for a given spi-logic.

In this paper, we investigate connections between various consequence relations for the fragment of propositional multi-modal logic that comprises implications $\sigma \to \tau$, where σ and τ are *strictly positive modal formulas* [8] constructed from propositional variables using conjunction $\land$, unary diamond operators $\Diamond_i$, and the constant 'truth' $\top$. We call such formulas σ and τ *sp-formulas* and implications between them *sp-implications*.

§1. Background.

Consequence relations for sp-implications have been studied in knowledge representation, universal algebra, and modal provability logic.

1.1. Description logic $\mathcal{EL}$. In knowledge representation, ontologies are used to define vocabularies for domains of interest together with logical relationships between the vocabulary terms [4, 56, 5]. The description logic $\mathcal{EL}$ [6, 3] is a widely used ontology language, in which such relationships are given by means of (notational variants of) sp-implications. A typical example of an $\mathcal{EL}$ ontology is SNOMED CT [67] that provides a standardised medical vocabulary for the healthcare systems of more than twenty countries. SNOMED CT consists of about 300,000 sp-implications covering most aspects of medicine and healthcare. For example, the sp-implication

$$\mathit{Viral\_pneumonia} \to \Diamond_{\mathit{causative\_agent}} \mathit{Virus} \land \Diamond_{\mathit{finding\_site}} \mathit{Lung}$$

says that viral pneumonia is caused by a virus and found in lungs. $\mathcal{EL}$ is the logical underpinning of the profile *OWL 2 EL* of the Web Ontology Language *OWL 2* [60] designed by W3C for writing up ontologies. Under the $\mathcal{EL}$ semantics, sp-implications are interpreted in relational structures known as Kripke frames in modal logic. Important reasoning problems are whether an sp-implication is valid under this semantics and, more generally, whether it follows from a finite set of sp-implications. The former is called the subsumption problem, its generalisation is the subsumption problem relative to a TBox. In modal logic, they correspond to the local and, respectively, global consequence relation (restricted to sp-implications). The computational complexity of these problems has been extensively studied. Both were shown to be PTIME-complete in general [6, 3] as well as under additional relational constraints and extensions to the language [3, 71], for example, over transitive Kripke frames and, more generally, frames satisfying implications of the form $R_1 \circ \cdots \circ R_n \subseteq R$, for binary relations $R_1, \ldots, R_n, R$. PTIME/coNP dichotomy results for the subsumption problem under some universally first-order definable relational constraints were obtained in [54], while [2] gave an example of a constraint under which subsumption becomes undecidable.

1.2. Semilattices with monotone operators. Following the algebraic approach to giving semantics to propositional logics [62], we can regard strictly positive modal formulas as terms of the algebraic language with a binary function $\wedge$, unary functions $\Diamond_i$ and constant $\top$. If $\wedge$ is a semilattice operation, then an sp-implication $\sigma \to \tau$ becomes an 'inequality' of the form $\sigma \leq \tau$, which is equationally expressible as $\sigma \wedge \tau \approx \sigma$. Conversely, any algebraic equation $\sigma \approx \tau$ between strictly positive 'terms' is equivalent to the pair $\sigma \to \tau$ and $\tau \to \sigma$ of sp-implications. Thus, semilattices with additional operators provide another natural semantics for sp-implications.

Semilattices with operators have been studied in universal algebra. An important example is their use in McKenzie's undecidability proof for Tarski's finite basis problem [57]. There has been extensive research on generalising natural dualities for algebras with various kinds of (semi)lattice reducts to algebras with operators [61, 76, 40, 1, 34, 43, 38, 35, 68, 33, 36, 26].

The relational semantics for the description logic $\mathcal{EL}$ mentioned above has been connected to the uniform word problem (aka quasiequational theory) of varieties of *semilattices with monotone*[1] *unary operators* (SLOs, for short) in [70, 71]. Varieties of closure semilattices, that is, SLOs with a single operator $\Diamond$ validating $p \leq \Diamond p$ and $\Diamond\Diamond p \leq \Diamond p$, have been investigated in [46]. They are also connected to the closure algebras of McKinsey and Tarski [58].

1.3. Sub-propositional modal logics and Reflection Calculus RC. Sp-implications have also been investigated in the context of provability logic [7, 25, 8, 11, 10]. The main motivation for considering them was the observation that, while syntactical modal reasoning in Japaridze's multi-modal provability logic GLP [48, 16] cannot be characterised by any class of Kripke frames, its restriction RC to sp-implications does have such a characterisation [25]. In particular, sp-implications are regarded in RC as sequents connecting two strictly positive formulas, and the developed syntactic calculus mimics the algebraic SLO-axioms and the axioms and rules of Birkhoff's equational calculus [12] (see §3.3 below). Note also that RC allows more general arithmetic interpretations than GLP [8] and, similarly to the subsumption problem in $\mathcal{EL}$, reasoning in RC is PTIME-complete [25] (whereas GLP is PSPACE-complete [65]).

[1] A unary operator $\Diamond_i$ in an algebra $\mathfrak{A}$ is called *monotone* if $\mathfrak{A}$ validates $\Diamond_i(p \wedge q) \leq \Diamond_i q$. This is the same as to say that $a \leq b$ implies $\Diamond_i a \leq \Diamond_i b$, for any a, b in $\mathfrak{A}$.

Other sub-propositional fragments of full modal logic that contain sp-formulas have also been considered in the literature, both in the modal and description logic setting and under various relational constraints. For example, results on the computational complexity of the fragment with formulas built from literals using $\wedge$ and both diamond and box modalities can be found in [64, 28, 44]. The above mentioned dualities have also been investigated from the modal logic perspective in order to find extensions of Kripke semantics that match the corresponding algebraic semantics; see [29, 17, 18, 69] for the negation-free fragment and [37] for its extension with $\wedge/\vee$-swapping operators.

In this paper, our concern is somewhat 'orthogonal' to duality theory: instead of modifying/extending the relational semantics to 'match' it with the algebraic one, we aim to understand the relationship between the (often intended) relational and (syntactic) algebraic consequence relations for sp-implications.

§2. Research problems and results. Following the modal logic tradition, we define the *spi-logic axiomatised by a set Σ of spi-implications* as the closure of Σ under the axioms and rules of a syntactic calculus capturing the algebraic semantics of sp-implications. We denote this logic by $L = \mathsf{SPi} + \Sigma$, indicating that SPi comprises the sp-implications that are valid in all SLOs.

Our primary concern is the (Kripke) completeness problem for spi-logics. More precisely, we would like to

(*completeness*): identify spi-logics $\mathsf{SPi} + \Sigma$ that are *complete* in the sense that the two consequence relations $\Sigma \models_{\mathsf{Kr}}$ and $\Sigma \models_{\mathsf{SLO}}$ coincide, where for any sp-implication $\boldsymbol{\iota}$,

$$\Sigma \models_{\mathsf{Kr}} \boldsymbol{\iota} \quad \text{iff} \quad \boldsymbol{\iota} \text{ is valid in every Kripke frame validating } \Sigma;$$
$$\Sigma \models_{\mathsf{SLO}} \boldsymbol{\iota} \quad \text{iff} \quad \boldsymbol{\iota} \text{ is valid in every SLO validating } \Sigma.$$

Sp-implications are modal Sahlqvist formulas [63]. So, by the completeness part of Sahlqvist's theorem, the full Boolean normal modal logic $\mathsf{K} \oplus \Sigma$ axiomatised (using the standard calculus of normal modal logic[2]) by the sp-implications in Σ is Kripke complete, that is, for every modal formula φ,

$$(1) \quad \Sigma \models_{\mathsf{Kr}} \varphi \;\text{ iff }\; \varphi \in \mathsf{K} \oplus \Sigma \;\text{ iff }\; \varphi \approx \top \text{ is valid in every BAO validating } \Sigma,$$

where BAO stands for *Boolean algebra with normal and $\vee$-additive unary operators*[3] [49]. Note that, by (1), the completeness problem is equivalent to

(*spi-axiomatisability*): the problem whether Σ *spi-axiomatises* the spi-fragment of the modal logic $\mathsf{K} \oplus \Sigma$, that is, $\boldsymbol{\iota} \in \mathsf{SPi} + \Sigma$ iff $\boldsymbol{\iota} \in \mathsf{K} \oplus \Sigma$, for any sp-implication $\boldsymbol{\iota}$ (in other words, the problem whether the spi-logic $\mathsf{SPi} + \Sigma$ has a *modal companion* [11]); and also to

(*conservativity*): the purely algebraic problem of whether the consequence relation $\Sigma \models_{\mathsf{BAO}}$ is *conservative* over $\Sigma \models_{\mathsf{SLO}}$ with respect to algebraic equations between sp-formulas, that is, $\Sigma \models_{\mathsf{SLO}} \sigma \approx \tau$ iff $\Sigma \models_{\mathsf{BAO}} \sigma \approx \tau$, for any sp-formulas σ and τ.

[2] It has the modal axioms $\Box_i(\varphi \to \psi) \to (\Box_i\varphi \to \Box_i\psi)$ and the rules of substitution, modus ponens and necessitation $\varphi/\Box_i\varphi$, for each modal operator $\Box_i$.

[3] A BAO is an algebra of the form $\mathfrak{A} = (A, \wedge, \vee, -, \bot, \top, \Diamond_i)_{i \in I}$, where $(A, \wedge, \vee, -, \bot, \top)$ is a Boolean algebra, $\Diamond_i\bot = \bot$ and $\Diamond_i(a \vee b) = \Diamond_i a \vee \Diamond_i b$, for all $a, b \in A$ and $i \in I$.

In Boolean modal logic, the completeness problem has been actively and thoroughly investigated since the invention of the Kripke semantics in the 1950–60s. Nearly all standard modal logics were proved to be Kripke complete by showing that they either are canonical or have the finite model property, and it took a while to *construct* first examples of incomplete logics [32, 73]. In contrast, incomplete spi-logics are easy to find, with two simplest ones being $\mathsf{SPi}+\{\Diamond p \to p\}$ and $\mathsf{SPi}+\{\Diamond p \to \Diamond q\}$ (Examples 1 and 2). It is readily seen that both of them have the finite model (but not finite frame) property. By Sahlqvist's theorem, all Boolean modal logics with sp-implicational axioms are canonical. Thus, the classical completeness theory appears to be of little help in understanding completeness of spi-logics. New tools and techniques are required to investigate this phenomenon.

In this paper, we develop and apply two general methods for establishing completeness of spi-logics.

The first one is based on the fact that an spi-logic L is complete whenever every SLO validating L can be embedded into the (SLO-reduct of the) full complex algebra of some Kripke frame for L. Following the terminology of Goldblatt [40], we call such spi-logics L *complex*. Proving that L is complex can be regarded as a generalisation of the canonical model technique from modal logic: for every BAO $\mathfrak{A}$ validating an spi-logic L, its ultrafilter-frame $\mathfrak{A}_+$ validates L as well. Unfortunately, no such 'canonical' Kripke frame construction is available for SLOs. Instead, we suggest two 'templates' that provide a range of embeddings of SLOs into the SLO-reducts of complex algebras of appropriate frames, one generalising the embedding of [46], and another one using filters in SLOs (see §4.1). We employ these templates to obtain two general sufficient conditions for complexity (and so completeness) of spi-logics (Theorems 19 and 35), and also show complexity of numerous concrete spi-logics defining familiar classes of Kripke frames. Our conditions cover earlier results of Sofronie-Stokkermans [70, 71] who proved that sp-implications of the form $\Diamond_1 \dots \Diamond_n p \to \Diamond_0 p$ axiomatise complex spi-logics, and those of Jackson [46] who showed that the spi-logic $\mathsf{SPi}_{qo} = \mathsf{SPi} + \{p \to \Diamond p, \Diamond\Diamond p \to \Diamond p\}$ (whose axioms $\Sigma_{qo} = \{p \to \Diamond p, \Diamond\Diamond p \to \Diamond p\}$ define the class of all quasiorders—frames of the modal logic $\mathsf{S4}$) is complex. We delimit the scope of the method by providing many examples of incomplete spi-logics, in particular, pairs of complete and incomplete spi-logics sharing the same Kripke frames, and develop a general technique for constructing incomplete spi-logics (Theorem 27).

As mentioned above, Boolean modal logics with sp-implicational axioms are always complex. In contrast, we show a few natural and simple sp-implications that axiomatise complete but not complex spi-logics, for example, those expressing n-functionality, for $n \geq 2$, and linearity (Theorems 39 and 47). For such spi-logics, we develop another general technique, called the method of *syntactic proxies*, that mimics Kripke frame reasoning with the help of the syntactic Birkhoff-type calculus for SLOs (see §4.2). We use this method to prove one more general sufficient condition for completeness (Theorem 20) and apply it to a number of concrete spi-logics that are not complex (Theorems 40, 41, 48). Syntactic proxies can also be used to establish completeness of all but two proper extensions of the spi-logic $\mathsf{SPi}_{equiv} = \mathsf{SPi} + \{p \to \Diamond p, \Diamond\Diamond p \to \Diamond p, q \wedge \Diamond p \to \Diamond(p \wedge \Diamond q)\}$

(whose axioms define the class of all equivalence relations—frames of the modal logic $\mathsf{S5}$), the two exceptions being in fact incomplete. Jackson [46] fully described the lattice of extensions of SPi_{equiv}; it follows from his proofs that most of them are $\models_{\mathsf{BAO}}$–to–$\models_{\mathsf{SLO}}$ conservative.

One feature that spi-logics do share with Boolean modal logics is that—apart from a few simple cases (such as extensions of SPi_{equiv} and $\mathsf{S5}$)—complete and effective classifications of logics according to their non-trivial properties are hardly possible. In §8, we prove by reduction of the halting problem for Turing machines that, given a finite set Σ of sp-implications, no algorithm can recognise completeness or complexity of the spi-logic $\mathsf{SPi} + \Sigma$. The proof is more direct compared to the known constructions from modal logic [74, 21, 19] because very simple incomplete spi-logics are available.

Having laid foundations for a completeness theory in the strictly positive context, we are naturally interested in the byproducts it may have for two related problems, viz., the computational complexity (in particular, decidability) of spi-logics and the definability problem. Recall that tractability of reasoning was one of the main motivations for considering spi-logics.

As far as ***computational complexity*** is concerned, we observe that spi-logics with universally definable classes of Kripke frames have the polynomial finite frame property[4] and are decidable in coNP if finitely axiomatisable and complete (Theorem 11); moreover, those complete ones whose frames are definable by equality-free universal Horn sentences are actually tractable (Theorem 13). The latter applies to the spi-logics in the scope of completeness Theorems 19, 20 and 23. (Note that Boolean modal logics axiomatised by the same sp-implications can be computationally very complex, even undecidable [52]). We also show tractability of several finitely axiomatisable complete spi-logics defining universal *non*-Horn frame conditions such as the spi-logic SPi^n_{equiv} whose frames are equivalence relations with classes of size $\leq n$, for $n \geq 2$ (Theorem 42), and the spi-fragment SPi_{lin} of the modal logic $\mathsf{S4.3}$ (Theorem 49). On the other hand, we observe that the completeness criterion of Theorem 35 has the spi-fragments of all modal grammar logics [30] in its scope, and so there exist finitely axiomatisable and undecidable complete spi-logics [75, 66, 20, 2, 11].

A class $\mathcal{C}$ of Kripke frames is called ***spi-definable*** if $\mathcal{C} = \{\mathfrak{F} \mid \mathfrak{F} \models \Sigma\}$ for some set Σ of sp-implications. The correspondence part of Sahlqvist's theorem [63] says that spi-definability (unlike modal definability) always implies definability by first-order $\forall\exists$-sentences. Many standard properties of frames turn out to be spi-definable (see Table 1). On the other hand, such well-known logics as $\mathsf{K4.1}$, $\mathsf{K4.2}$ and $\mathsf{K4.3}$ are typical examples of Kripke complete modal logics whose frames are not spi-definable (see Table 2). To obtain such non-spi-definability results, we give a general necessary condition for spi-definability (in §9.1), and also show that spi-definable properties of quasiorders must be universal.

The remainder of the article is organised as follows. Having defined in §3 the required basic notions, in §4 we introduce the two general methods for establishing completeness, which are applied in §§5–7 and complemented by multiple

[4] An spi-logic L has the *polynomial finite frame property* if every sp-implication $\boldsymbol{\iota}$ that fails in some frame for L also fails in a frame for L of polynomial size in $\boldsymbol{\iota}$.

examples of incomplete spi-logics. We systematise our completeness results for spi-logics according to the form of the first-order correspondents of their axioms: sp-implications with universal Horn, existential and disjunctive correspondents are discussed in §5, §6 and §7, respectively. In §8 we prove that it is undecidable whether a given finite set of sp-implications axiomatises a complete or complex spi-logic. A few related problems are briefly discussed in §9: in §9.1 we deal with non-spi-definability; in §9.2 we consider sp$^\perp$-implications that may also contain the constant $\perp$ standing for 'falsehood' in Kripke frames and for the $\leq$-smallest element in SLOs; in §9.3 we have a brief look at *spi-rule logics* (quasiequational theories in the algebraic setting). In particular, we characterise complex spi-rule logics rL as those for which $rL \models_{\mathsf{Kr}} \boldsymbol{\rho}$ coincides with $rL \models_{\mathsf{SLO}} \boldsymbol{\rho}$, for all spi-rules $\boldsymbol{\rho}$. Finally, in §10 we suggest further research directions; a few open questions are also scattered throughout the paper.

TABLE 1. Spi-definable first-order properties.

first-order property	sp-implication(s)	notation
reflexivity	$p \to \Diamond p$	$\boldsymbol{\iota}_{\mathit{refl}}$
transitivity	$\Diamond\Diamond p \to \Diamond p$	$\boldsymbol{\iota}_{\mathit{trans}}$
symmetry	$q \wedge \Diamond p \to \Diamond(p \wedge \Diamond q)$	$\boldsymbol{\iota}_{\mathit{sym}}$
$\forall x, y, z \,\big(R(x,y) \wedge R(x,z) \to R(y,z)\big)$ Euclideanness	$\Diamond p \wedge \Diamond q \to \Diamond(p \wedge \Diamond q)$	$\boldsymbol{\iota}_{\mathit{eucl}}$
quasiorder	$\{\boldsymbol{\iota}_{\mathit{refl}}, \boldsymbol{\iota}_{\mathit{trans}}\}$	Σ_{qo}
equivalence	$\{\boldsymbol{\iota}_{\mathit{refl}}, \boldsymbol{\iota}_{\mathit{trans}}, \boldsymbol{\iota}_{\mathit{sym}}\}$ $\{\boldsymbol{\iota}_{\mathit{refl}}, \boldsymbol{\iota}_{\mathit{trans}}, \boldsymbol{\iota}_{\mathit{eucl}}\}$	Σ_{equiv} Σ'_{equiv}
$\forall x, y, z \,\big[R(x,y) \wedge R(x,z) \to \big(R(y,y) \wedge R(y,z)\big) \vee \big(R(z,z) \wedge R(z,y)\big)\big]$	$\Diamond(p \wedge q) \wedge \Diamond(p \wedge r) \to \Diamond(p \wedge \Diamond q \wedge \Diamond r)$	$\boldsymbol{\iota}_{\mathit{wcon}}$
linear quasiorder[5]	$\{\boldsymbol{\iota}_{\mathit{refl}}, \boldsymbol{\iota}_{\mathit{trans}}, \boldsymbol{\iota}_{\mathit{wcon}}\}$	Σ_{lin}
$\forall x, y \,\big[R(x,y) \to \exists z \,\big(R(x,z) \wedge R(z,y)\big)\big]$ density	$\Diamond p \to \Diamond\Diamond p$	$\boldsymbol{\iota}_{\mathit{dense}}$
$\forall x, y, z \,\big(R(x,y) \wedge R(x,z) \to (y = z)\big)$ functionality	$\Diamond p \wedge \Diamond q \to \Diamond(p \wedge q)$	$\boldsymbol{\iota}_{\mathit{fun}}$

§3. Preliminaries. We begin by giving definitions of the basic notions and discussing the problems we deal with in this paper.

3.1. Sp-formulas and sp-implications. Let $\mathcal{R}$ be a non-empty set called a *signature*. An *sp-formula* (*of signature* $\mathcal{R}$) is a multi-modal formula constructed from propositional variables p from some countably infinite set var and constant $\top$ using conjunction $\wedge$ and unary diamond operators $\Diamond_R$, for $R \in \mathcal{R}$. We omit the subscript R in the unimodal case $\mathcal{R} = \{R\}$.

An *sp-implication* $\boldsymbol{\iota}$ (*of signature* $\mathcal{R}$) is an expression of the form $\sigma \to \tau$, where σ and τ are sp-formulas of signature $\mathcal{R}$.

[5] A reflexive and transitive relation R is called a *linear quasiorder* if R is *weakly connected*: $\forall x, y, z \,\big(R(x,y) \wedge R(x,z) \to R(y,z) \vee R(z,y) \vee (y = z)\big)$. Linear quasiorders are the frames of the modal logic $\mathsf{S4.3}$.

TABLE 2. Non-spi-definable but modally definable first-order properties.

first-order property	modal formula(s)	notation
$\forall x, y, z\, \big(R(x,y) \wedge R(y,z) \to R(x,z) \vee (x = z)\big)$ pseudo-transitivity	$\Diamond\Diamond p \to p \vee \Diamond p$	φ_{ptrans}
pseudo-equivalence	$\boldsymbol{\iota}_{sym},\ \varphi_{ptrans}$	Diff
weak connectedness[5]	$\Diamond p \wedge \Diamond q \to \Diamond(p \wedge q) \vee \Diamond(p \wedge \Diamond q) \vee \Diamond(q \wedge \Diamond p)$	φ_{wcon}
transitivity and weak connectedness	$\boldsymbol{\iota}_{trans},\ \varphi_{wcon}$	K4.3
$\forall x, y, z\, \big(R(x,y) \wedge R(x,z) \to \exists u\, \big(R(y,u) \wedge R(z,u)\big)\big)$ confluence	$\Diamond\Box p \to \Box\Diamond p$	φ_{conf}
transitivity and confluence	$\boldsymbol{\iota}_{trans},\ \varphi_{conf}$	K4.2
transitivity and $\forall x\, \exists y \big(R(x,y) \wedge \forall z\, \big(R(y,z) \to (y = z)\big)\big)$	$\boldsymbol{\iota}_{trans},\ \Box\Diamond p \to \Diamond\Box p$	K4.1

As argued in §§1–2, we aim to connect two types of semantics for sp-implications: one based on first-order relational structures, known as Kripke frames in modal logic, and an algebraic one, based on meet-semilattices with monotone operators. We begin with the latter.

3.2. Algebraic semantics. A structure $\mathfrak{A} = (A, \wedge, \top, \Diamond_R)_{R \in \mathcal{R}}$ is an *sp-type algebra* (*of signature* $\mathcal{R}$) if $A \neq \emptyset$, $\top \in A$, $\wedge$ is a binary and each $\Diamond_R$ a unary function (operator) on A. This way sp-formulas can be regarded as algebraic *sp-type terms*. (The overloading of $\wedge$, $\top$ and $\Diamond_R$ should not confuse the reader as it will always be clear from context whether we deal with algebraic operations or logic connectives.) An *sp-type equation* is of the form $\sigma \approx \tau$, where σ and τ are sp-type terms (that is, sp-formulas). A *valuation in* $\mathfrak{A}$ is a function $\mathfrak{a}$ mapping the variables $p \in \mathsf{var}$ to elements in A. The *value* $\tau[\mathfrak{a}] \in A$ of an sp-type term τ under $\mathfrak{a}$ is defined inductively as usual. If the variables occurring in τ are among $p_1, \dots, p_n$ and $\mathfrak{a}(p_i) = a_i$, then we also write $\tau[a_1, \dots, a_n]$ in place of $\tau[\mathfrak{a}]$. Given an sp-type equation $\sigma \approx \tau$, we set $\mathfrak{A} \models (\sigma \approx \tau)[\mathfrak{a}]$ if $\sigma[\mathfrak{a}] = \tau[\mathfrak{a}]$, and $\mathfrak{A} \models (\sigma \approx \tau)$ if $\mathfrak{A} \models (\sigma \approx \tau)[\mathfrak{a}]$ for every valuation $\mathfrak{a}$ in $\mathfrak{A}$, in which case we say that $\mathfrak{A}$ *validates* $\sigma \approx \tau$.

A *meet-semilattice with monotone operators* (SLO, for short) is an sp-type algebra validating the following sp-type equations:

$$p \wedge p \approx p, \tag{2}$$

$$p \wedge q \approx q \wedge p, \tag{3}$$

$$p \wedge (q \wedge r) \approx (p \wedge q) \wedge r, \tag{4}$$

$$p \wedge \top \approx p, \tag{5}$$

$$\Diamond_R (p \wedge q) \wedge \Diamond_R q \approx \Diamond_R (p \wedge q), \quad \text{for } R \in \mathcal{R}. \tag{6}$$

In a SLO $\mathfrak{A}$, the partial order $\leq$ is defined as usual by taking $a \leq b$ iff $a \wedge b = a$, for all a, b in $\mathfrak{A}$. It is readily seen that $\wedge$ and $\Diamond_R$ are *monotone* with respect to $\leq$: if $a \leq b$ then $a \wedge c \leq b \wedge c$ and $\Diamond_R a \leq \Diamond_R b$, for all a, b, c in $\mathfrak{A}$ and $R \in \mathcal{R}$.

By regarding any sp-implication $\boldsymbol{\iota} = (\sigma \to \tau)$ as an sp-type 'inequality' $\sigma \leq \tau$ (which is a shorthand for the sp-type equation $\sigma \wedge \tau \approx \sigma$), we set $\mathfrak{A} \models \boldsymbol{\iota}[\mathfrak{a}]$ if $\sigma[\mathfrak{a}] \leq \tau[\mathfrak{a}]$, and $\mathfrak{A} \models \boldsymbol{\iota}$ if $\mathfrak{A} \models \boldsymbol{\iota}[\mathfrak{a}]$ for every valuation $\mathfrak{a}$ in $\mathfrak{A}$, in which case we say that $\mathfrak{A}$ *validates* $\boldsymbol{\iota}$. The set of sp-implications that are validated by all SLOs is denoted by SPi.

We say that a SLO $\mathfrak{A}$ *validates* a set Σ of sp-implications and write $\mathfrak{A} \models \Sigma$ if $\mathfrak{A} \models \boldsymbol{\iota}$ for all $\boldsymbol{\iota}$ in Σ. We denote by SLO_Σ the class—in fact, variety—of all SLOs validating Σ. In particular, SLO denotes the variety of all SLOs. We define a consequence relation $\Sigma \models_{\mathsf{SLO}}$ by taking, for any sp-implication $\boldsymbol{\iota}$,

$$\Sigma \models_{\mathsf{SLO}} \boldsymbol{\iota} \quad \text{iff} \quad \mathfrak{A} \models \boldsymbol{\iota} \ \text{ for every } \mathfrak{A} \in \mathsf{SLO}_\Sigma.$$

We write $\models_{\mathsf{SLO}} \boldsymbol{\iota}$ for $\emptyset \models_{\mathsf{SLO}} \boldsymbol{\iota}$. As a SLO clearly validates $\sigma \approx \tau$ iff it validates both $\sigma \to \tau$ and $\tau \to \sigma$, we write $\Sigma \models_{\mathsf{SLO}} \sigma \approx \tau$ whenever both $\Sigma \models_{\mathsf{SLO}} \sigma \to \tau$ and $\Sigma \models_{\mathsf{SLO}} \tau \to \sigma$ hold.

3.3. Spi-logics. As sp-implications are special cases of algebraic sp-type equations, the consequence relation $\Sigma \models_{\mathsf{SLO}}$ can be characterised syntactically by Birkhoff's equational calculus [12, 42]. Using a Lindenbaum–Tarski-algebra type argument, it is readily seen that $\Sigma \models_{\mathsf{SLO}}$ can also be captured by a calculus using only sp-implications in its derivations. Namely, it is not hard to show that

$$\Sigma \models_{\mathsf{SLO}} \boldsymbol{\iota} \quad \text{iff} \quad \Sigma \vdash_{\mathsf{SLO}} \boldsymbol{\iota}, \tag{7}$$

where $\Sigma \vdash_{\mathsf{SLO}} \boldsymbol{\iota}$ means that there is a finite sequence $\boldsymbol{\iota}_0, \dots, \boldsymbol{\iota}_n$ of sp-implications such that $\boldsymbol{\iota}_n = \boldsymbol{\iota}$ and each $\boldsymbol{\iota}_i$, for $i \leq n$, is either a substitution instance of some sp-implication in Σ or a substitution instance of one of the axioms

$$p \to p, \qquad p \to \top, \qquad p \wedge q \to q \wedge p, \qquad p \wedge q \to p, \tag{8}$$

or obtained from earlier members of the sequence using one of the rules

$$\frac{\sigma \to \tau \quad \tau \to \varrho}{\sigma \to \varrho}, \qquad \frac{\sigma \to \tau \quad \sigma \to \varrho}{\sigma \to \tau \wedge \varrho}, \qquad \frac{\sigma \to \tau}{\Diamond_R \sigma \to \Diamond_R \tau} \quad (R \in \mathcal{R}) \tag{9}$$

(see also the Reflection Calculus **RC** of [7, 25]). In fact, throughout we shall only use the $\Leftarrow$ (soundness) direction of (7). We write $\vdash_{\mathsf{SLO}} \boldsymbol{\iota}$ for $\emptyset \vdash_{\mathsf{SLO}} \boldsymbol{\iota}$. We write $\Sigma \vdash_{\mathsf{SLO}} \sigma \approx \tau$ whenever both $\Sigma \vdash_{\mathsf{SLO}} \sigma \to \tau$ and $\Sigma \vdash_{\mathsf{SLO}} \tau \to \sigma$ hold.

For any set Σ of sp-implications, we define the *spi-logic* $\mathsf{SPi} + \Sigma$ *axiomatised by* Σ as

$$\mathsf{SPi} + \Sigma \ = \ \{\boldsymbol{\iota} \mid \boldsymbol{\iota} \text{ is an sp-implication and } \Sigma \vdash_{\mathsf{SLO}} \boldsymbol{\iota}\}.$$

If $L = \mathsf{SPi} + \Sigma$, for some set Σ of sp-implications, then we call L an *spi-logic*.

3.4. Kripke semantics. A *Kripke model* (*of signature* $\mathcal{R}$) is a pair of the form $\mathfrak{M} = (\mathfrak{F}, \mathfrak{v})$, where $\mathfrak{F} = (W, R^{\mathfrak{F}})_{R \in \mathcal{R}}$ is a *frame* (*of signature* $\mathcal{R}$) with domain $W \neq \emptyset$ and binary (accessibility) relations $R^{\mathfrak{F}}$, for $R \in \mathcal{R}$, and $\mathfrak{v}$ is a *valuation* associating a subset $\mathfrak{v}(p) \subseteq W$ with any variable p. The truth relation $\mathfrak{M}, w \models \tau$ for $w \in W$ and an sp-formula τ is defined by induction: $\mathfrak{M}, w \models \top$, $\mathfrak{M}, w \models p$ iff $w \in \mathfrak{v}(p)$, $\mathfrak{M}, w \models \tau' \wedge \tau''$ iff $\mathfrak{M}, w \models \tau'$ and $\mathfrak{M}, w \models \tau''$, and for each $R \in \mathcal{R}$,

$$\mathfrak{M}, w, \models \Diamond_R \tau' \quad \text{iff} \quad \mathfrak{M}, w' \models \tau' \text{ for some } w' \text{ with } (w, w') \in R^{\mathfrak{F}}.$$

For an sp-implication $\boldsymbol{\iota} = (\sigma \to \tau)$ and $w \in W$, we write $\mathfrak{M}, w \models \boldsymbol{\iota}$ if $\mathfrak{M}, w \models \sigma$ implies $\mathfrak{M}, w \models \tau$. We say that $\boldsymbol{\iota}$ *holds* in $\mathfrak{M}$ (or $\mathfrak{M}$ is a *model of* $\boldsymbol{\iota}$) and write

$\mathfrak{M} \models \boldsymbol{\iota}$, if $\mathfrak{M}, w \models \boldsymbol{\iota}$ holds for every $w \in W$. We also write $\mathfrak{F}, w \models \boldsymbol{\iota}$ if $\mathfrak{M}, w \models \boldsymbol{\iota}$ holds for every Kripke model $\mathfrak{M}$ based on $\mathfrak{F}$, and $\mathfrak{F} \models \boldsymbol{\iota}$ if $\mathfrak{F}, w \models \boldsymbol{\iota}$ for every $w \in W$ (equivalently, if $\mathfrak{M} \models \boldsymbol{\iota}$ for every model $\mathfrak{M}$ based on $\mathfrak{F}$); in this case, we say that $\mathfrak{F}$ *validates* $\boldsymbol{\iota}$. Finally, we say that $\mathfrak{F}$ *validates* (or is a *frame for*) a set Σ of sp-implications and write $\mathfrak{F} \models \Sigma$, if $\mathfrak{F} \models \boldsymbol{\iota}$ for every $\boldsymbol{\iota}$ in Σ. The class of frames for Σ is denoted by Kr_Σ. By the correspondence part of Sahlqvist's theorem, Kr_Σ is first-order definable in the language with binary predicate symbols R, for $R \in \mathcal{R}$, and equality. Any such first-order theory defining Kr_Σ is called a *correspondent of* Σ; see, e.g., [13, 22]. (All correspondents of Σ are equivalent.) If $\{\Psi\}$ is a correspondent of $\{\boldsymbol{\iota}\}$, we say that Ψ is a *correspondent of* $\boldsymbol{\iota}$.

Given a set Σ of sp-implications, we define a consequence relation $\Sigma \models_{\mathsf{Kr}}$ by taking, for any sp-implication $\boldsymbol{\iota}$,

$$\Sigma \models_{\mathsf{Kr}} \boldsymbol{\iota} \quad \text{iff} \quad \mathfrak{F} \models \boldsymbol{\iota} \text{ for every frame } \mathfrak{F} \in \mathsf{Kr}_\Sigma.$$

We write $\models_{\mathsf{Kr}} \boldsymbol{\iota}$ for $\emptyset \models_{\mathsf{Kr}} \boldsymbol{\iota}$.

3.5. Completeness. Every frame $\mathfrak{F} = (W, R^{\mathfrak{F}})_{R \in \mathcal{R}}$ gives rise to a SLO

$$\mathfrak{F}^\star = (2^W, \cap, W, \Diamond_R^+)_{R \in \mathcal{R}},$$

where, for all $R \in \mathcal{R}$ and $X \subseteq W$,

$$\Diamond_R^+ X = \{w \in W \mid (w, v) \in R^{\mathfrak{F}} \text{ for some } v \in X\}$$

(that is, $\mathfrak{F}^\star$ is the sp-type reduct of the *full complex algebra of* $\mathfrak{F}$ [40]). As Kripke models over $\mathfrak{F}$ and valuations in $\mathfrak{F}^\star$ are the same thing, for every sp-implication $\boldsymbol{\iota}$, we have $\mathfrak{F} \models \boldsymbol{\iota}$ iff $\mathfrak{F}^\star \models \boldsymbol{\iota}$. Therefore, for every spi-logic $\mathsf{SPi} + \Sigma$,

$$\Sigma \models_{\mathsf{SLO}} \boldsymbol{\iota} \quad \Longrightarrow \quad \Sigma \models_{\mathsf{Kr}} \boldsymbol{\iota}, \quad \text{for any } \boldsymbol{\iota}, \tag{10}$$

and so, by (7),

$$\mathsf{Kr}_\Sigma = \mathsf{Kr}_{\mathsf{SPi}+\Sigma}. \tag{11}$$

An spi-logic $L = \mathsf{SPi} + \Sigma$ is called *complete* if, for every sp-implication $\boldsymbol{\iota}$,

$$\Sigma \models_{\mathsf{Kr}} \boldsymbol{\iota} \quad \text{iff} \quad \Sigma \models_{\mathsf{SLO}} \boldsymbol{\iota}.$$

Note that completeness of L does not depend on its axioms: if $L = \mathsf{SPi} + \Sigma = \mathsf{SPi} + \Sigma'$ then $\mathsf{SLO}_L = \mathsf{SLO}_\Sigma = \mathsf{SLO}_{\Sigma'}$, and so $\mathsf{Kr}_L = \mathsf{Kr}_\Sigma = \mathsf{Kr}_{\Sigma'}$ by (11).

As discussed in §2, $\mathsf{SPi}+\emptyset$ and SPi_{qo} are simple examples of complete spi-logics [70, 46] (see also Theorem 4 and its proofs in §4.1 and §4.2, and Corollary 16). The following two examples show incomplete ones.

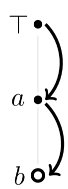
EXAMPLE 1. Consider the sp-implication $\Diamond p \to p$. On the one hand, a frame $\mathfrak{F} = (W, R^{\mathfrak{F}})$ validates $\Diamond p \to p$ iff $\mathfrak{F} \models \forall x, y\, (R(x, y) \to (x = y))$. Thus, it is easy to see that $\{\Diamond p \to p\} \models_{\mathsf{Kr}} \boldsymbol{\iota}$, where $\boldsymbol{\iota} = (p \wedge \Diamond \top \to \Diamond p)$. On the other hand, $\{\Diamond p \to p\} \not\models_{\mathsf{SLO}} \boldsymbol{\iota}$ as the SLO $\mathfrak{A}$ with 3 elements $b \leq a \leq \top$ such that $\Diamond a = \Diamond b = b$ and $\Diamond \top = a$ validates $\Diamond p \to p$ and refutes $\boldsymbol{\iota}$, since $a \wedge \Diamond \top = a \neq b = \Diamond a$ (see Fig. 1 (a)). So, the spi-logic $\mathsf{SPi} + \{\Diamond p \to p\}$ is incomplete.

EXAMPLE 2. Consider the sp-implication $\Diamond p \to \Diamond q$. On the one hand, a frame $\mathfrak{F} = (W, R^{\mathfrak{F}})$ validates $\Diamond p \to \Diamond q$ iff $R^{\mathfrak{F}} = \emptyset$, and so $\{\Diamond p \to \Diamond q\} \models_{\mathsf{Kr}} \Diamond \top \to p$. On the other hand, $\{\Diamond p \to \Diamond q\} \not\models_{\mathsf{SLO}} \Diamond \top \to p$ as the SLO $\mathfrak{A}$ with two elements

$a \leq \top$ such that $\Diamond a = \Diamond\top = \top$ validates $\Diamond p \to \Diamond q$ and refutes $\Diamond\top \to p$, since $\Diamond\top = \top \not\leq a$. Therefore, the spi-logic $\mathsf{SPi} + \{\Diamond p \to \Diamond q\}$ is incomplete.

3.6. Drawing SLOs. In our examples, depending on the context, we depict SLOs in two different ways. One way is to represent the semilattice structure by its Hasse diagram and use arrows labelled by R to indicate the $\Diamond_R$ functions. In the unimodal case, we represent the elements x with $\Diamond x = x$ by hollow circles, and indicate $\Diamond$ by unlabelled arrows otherwise; see Fig. 1 (a).

Another way is to draw a SLO as a subalgebra $\mathfrak{A}$ of some suitable $\mathfrak{F}^\star$ (which always exists by Theorem 3). We represent the underlying $\mathfrak{F} = (W, R^{\mathfrak{F}})_{R\in\mathcal{R}}$ as a labelled directed multigraph (omitting the edge labels in the unimodal case) and indicate the non-empty subsets of W that belong to $\mathfrak{A}$. This representation makes it easier for the 'modal logic minded' reader to check whether the given SLO validates an sp-implication $\boldsymbol{\iota}$: it suffices to verify that $\mathfrak{M} \models \boldsymbol{\iota}$ for every $\mathfrak{A}$-*admissible* Kripke model $\mathfrak{M}$ based on $\mathfrak{F}$, in which all $\mathfrak{M}(p)$ belong to the indicated subsets of $\mathfrak{F}$ (cf. *general frames* in modal logic [39, 22]). In Fig. 1 (b), showing such a drawing of the SLO $\mathfrak{A}$ from Example 1, $\mathfrak{M} \models \Diamond p \to p$ for all $\mathfrak{A}$-admissible Kripke models over the depicted $\mathfrak{F}$ (the model $(\mathfrak{F}, \mathfrak{v})$ with $\mathfrak{v}(p) = \{2\}$ is not $\mathfrak{A}$-admissible), while $\mathfrak{M}', 1 \not\models p \wedge \Diamond\top \to \Diamond p$ for $\mathfrak{M}' = (\mathfrak{F}, \mathfrak{v}')$ with $\mathfrak{v}'(p) = \{1\}$.



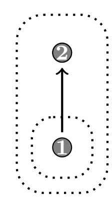


FIGURE 1. Two ways of depicting the SLO $\mathfrak{A}$ of Example 1.

§4. Tools and techniques for proving completeness. In this section, we introduce two general methods for proving completeness of spi-logics. Both methods will be illustrated by many examples throughout the paper.

4.1. Embedding SLOs into complex algebras of frames. Adopting the terminology of Goldblatt [40], we call an spi-logic L *complex* if every $\mathfrak{A}$ in SLO_L is embeddable[6] into $\mathfrak{F}^\star$, for some frame $\mathfrak{F}$ for L. As sp-implications are preserved under taking subalgebras, we always have that

$$L \text{ is complex} \quad\Longrightarrow\quad L \text{ is complete.}$$

Theorems 39 and 47 give examples where the converse implication does not hold.

It is well-known that every BAO is embeddable into the full complex algebra of its ultrafilter frame [49]. As shown in [70], a similar result also holds for SLOs:

[6] Given sp-type algebras $\mathfrak{A}$ and $\mathfrak{B}$ of the same signature, a function $\eta\colon \mathfrak{A} \to \mathfrak{B}$ is an *sp-homomorphism* if it preserves all the sp-operations. A one-to-one sp-homomorphism is an *sp-embedding*. $\mathfrak{A}$ is *embeddable into* $\mathfrak{B}$ if there exists an sp-embedding $\eta\colon \mathfrak{A} \to \mathfrak{B}$ (that is, if $\mathfrak{A}$ is isomorphic to a subalgebra of $\mathfrak{B}$). For universal algebra basics, we refer the reader to [42].

THEOREM 3. *Every* SLO *is embeddable into* $\mathfrak{F}^\star$*, for some frame* $\mathfrak{F}$*.*

As an immediate consequence, we obtain:

THEOREM 4. *The spi-logic* $\mathsf{SPi} + \emptyset$ *is complex, and so complete.*

The simple proposition below provides us with infinitely many complex spi-logics. Call an sp-implication $\sigma \to \tau$ *variable-free* if both σ and τ are built up from $\top$ using $\wedge$ and the $\Diamond_R$.

PROPOSITION 5. *If* $\mathsf{SPi}+\Sigma$ *is a complex spi-logic and* Σ_0 *a set of variable-free sp-implications, then* $\mathsf{SPi} + (\Sigma \cup \Sigma_0)$ *is complex.*

PROOF. By possibly adding 'dummy' sp-implications to Σ, we may assume that every $\Diamond_R$ occurring in Σ_0 also occurs in Σ. Suppose $\mathfrak{A} \in \mathsf{SLO}_{\Sigma\cup\Sigma_0}$. As $\mathsf{SPi} + \Sigma$ is complex, $\mathfrak{A}$ is (isomorphic to) a subalgebra of $\mathfrak{F}^\star$, for some frame $\mathfrak{F} \models \Sigma$. As $\mathfrak{A} \models \Sigma_0$, there is an $\mathfrak{A}$-admissible Kripke model $\mathfrak{M} \models \Sigma_0$ based on $\mathfrak{F}$. Since Σ_0 is variable-free, we also have $\mathfrak{F} \models \Sigma_0$. $\dashv$

QUESTION 1. Does Proposition 5 hold with 'complete' in place of 'complex'?

In the remainder of §4.1, we show two different ways of proving Theorem 3 and discuss connections between them.

4.1.1. *Embeddings via elements of SLOs.* These are variants of the embedding used by Jackson [46] for closure algebras. We embed a SLO $\mathfrak{A} = (A, \wedge, \top, \Diamond_R)_{R\in\mathcal{R}}$ into the SLO $\mathfrak{F}^\star$, for some frame $\mathfrak{F} = (A, R^{\mathfrak{F}})_{R\in\mathcal{R}}$, using the map

$$\eta\colon a \mapsto \{b \in A \mid b \leq a\}.$$

Clearly, $\eta(\top) = A$ and $\eta(a \wedge b) = \eta(a) \cap \eta(b)$. We show now that to preserve the $\Diamond_R$, it is enough if $R^{\mathfrak{F}}$ satisfies the following two conditions, for all $R \in \mathcal{R}$:

$$\forall a, b\ \big[(a, b) \in R^{\mathfrak{F}} \quad\Rightarrow\quad a \leq \Diamond_R\, b\big], \tag{12}$$

$$\forall a, b\ \big[a \leq \Diamond_R\, b \quad\Rightarrow\quad \exists c\,\big(c \leq b \text{ and } (a, c) \in R^{\mathfrak{F}}\big)\big]. \tag{13}$$

First we establish $\eta(\Diamond_R\, a) \subseteq \Diamond_R^+\eta(a)$. Let $b \leq \Diamond_R\, a$. By (13), there is $c \in A$ with $c \leq a$ and $(b, c) \in R^{\mathfrak{F}}$. It follows that $c \in \eta(a)$, and so $b \in \Diamond_R^+\eta(a)$. To show $\Diamond_R^+\eta(a) \subseteq \eta(\Diamond_R\, a)$, take any $b \in A$ such that $(b, x) \in R^{\mathfrak{F}}$ for some $x \in \eta(a)$. Then $x \leq a$ and, by (12), $b \leq \Diamond_R\, x$. By the monotonicity of $\Diamond_R$, $\Diamond_R\, x \leq \Diamond_R\, a$, and so $b \leq \Diamond_R\, a$, that is, $b \in \eta(\Diamond_R\, a)$. (In fact, it is easy to see that (12) and (13) are actually equivalent to $\forall a\, \eta(\Diamond_R\, a) = \Diamond_R^+\eta(a)$.) Finally, we check that η is injective. If $a, b \in A$ and $a \neq b$ then we may assume that $a \not\leq b$, in which case $a \in \eta(a)$ but $a \notin \eta(b)$.

For example, an $R^{\mathfrak{F}}$ satisfying (12) and (13) can be defined by taking

$$(a, b) \in R^{\mathfrak{F}} \quad\Longleftrightarrow\quad a \leq \Diamond_R\, b. \tag{14}$$

We use this definition in the proofs of Theorems 19 and 35. However, the proofs of Theorems 15, 23, 28 and 29 require different $R^{\mathfrak{F}}$ satisfying (12) and (13).

4.1.2. *Embeddings via filters.* Let $\mathfrak{A} = (A, \wedge, \top, \Diamond_R)_{R\in\mathcal{R}}$ be a SLO. For any $U \subseteq A$ and $R \in \mathcal{R}$, we set

$$\Diamond_R [U] = \{\Diamond_R a \mid a \in U\}.$$

We remind the reader that a nonempty subset $U \subseteq A$ is a *filter* (*of* $\mathfrak{A}$) if it is *up-closed* (in the sense that $a \in U$ and $a \leq b$ imply $b \in U$) and $\wedge$-*closed* (that is, $a \wedge b \in U$ for any $a, b \in U$). We denote by $\mathcal{F}(\mathfrak{A})$ the set of all filters of $\mathfrak{A}$.

We embed $\mathfrak{A}$ into $\mathfrak{G}^\star$, for some frame $\mathfrak{G} = (\mathcal{F}(\mathfrak{A}), R^{\mathfrak{G}})_{R\in\mathcal{R}}$, using the map

$$f\colon a \mapsto \{U \in \mathcal{F}(\mathfrak{A}) \mid a \in U\}.$$

Clearly, $f(\top) = \mathcal{F}(\mathfrak{A})$ and $f(a \wedge b) = f(a) \cap f(b)$ for all $a, b \in A$. Also, it is readily seen that to ensure $f\big(\Diamond_R a\big) = \Diamond_R^+ f(a)$ for all a, we can equivalently require that the following two conditions hold for all $U \in \mathcal{F}(\mathfrak{A})$ and $R \in \mathcal{R}$:

$$\forall V \ \big((U, V) \in R^{\mathfrak{G}} \quad\Rightarrow\quad \Diamond_R [V] \subseteq U\big), \tag{15}$$

$$\forall a \ \big[\Diamond_R a \in U \quad\Rightarrow\quad \exists V \big(a \in V \ \text{ and } \ (U, V) \in R^{\mathfrak{G}}\big)\big]. \tag{16}$$

To check that f is injective, let $a \neq b$. We may assume that $a \not\leq b$ and take the filter $\{a\}\!\uparrow = \{b \mid a \leq b\}$ (the *principal filter* generated by a). Then $\{a\}\!\uparrow \in f(a)$ but $\{a\}\!\uparrow \notin f(b)$.

For example, one can define $R^{\mathfrak{G}}$ by taking

$$(U, V) \in R^{\mathfrak{G}} \quad\Longleftrightarrow\quad \Diamond_R [V] \subseteq U. \tag{17}$$

Again, in general, there can be different $R^{\mathfrak{G}}$ satisfying (15) and (16); see, e.g., the proofs of Theorems 21 and 29 (*i*).

4.1.3. *Connection between the two embeddings.* For an arbitrary SLO $\mathfrak{A}$, with the 'classical' definitions of $R^{\mathfrak{F}}$ and $R^{\mathfrak{G}}$ via (14) and (17), respectively, we have the following:

PROPOSITION 6. *The frame* $(A, R^{\mathfrak{F}})_{R\in\mathcal{R}}$ *is isomorphic to a* (*not necessarily generated*) *subframe of* $(\mathcal{F}(\mathfrak{A}), R^{\mathfrak{G}})_{R\in\mathcal{R}}$. *For finite* $\mathfrak{A}$, *these frames are isomorphic.*

PROOF. For all $a, b \in A$, we have $(a, b) \in R^{\mathfrak{F}}$ iff $a \leq \Diamond_R b$ iff $a \leq \Diamond_R c$ for all $c \geq b$ iff $\big(\{a\}\!\uparrow, \{b\}\!\uparrow\big) \in R^{\mathfrak{G}}$. If $\mathfrak{A}$ is finite, then all filters of $\mathfrak{A}$ are principal. ⊣

4.2. Completeness by syntactic proxies. To introduce our second method for proving completeness, we establish some connections between sp-formulas and Kripke models.

Given Kripke models $\mathfrak{M}_i = (\mathfrak{F}_i, \mathfrak{v}_i)$ based on frames $\mathfrak{F}_i = (W_i, R_i)_{R\in\mathcal{R}}$, for $i = 1, 2$, a map $h\colon W_1 \to W_2$ is called an $\mathfrak{F}_1 \to \mathfrak{F}_2$ *homomorphism* if $(x, y) \in R_1$ implies $\big(h(x), h(y)\big) \in R_2$, for any $x, y \in W_1$ and $R \in \mathcal{R}$. If in addition $x \in \mathfrak{v}_1(p)$ implies $h(x) \in \mathfrak{v}_2(p)$, for any $x \in W_1$ and variable p, then h is called an $\mathfrak{M}_1 \to \mathfrak{M}_2$ *homomorphism*. Clearly, sp-formulas are preserved under homomorphisms in the sense that $\mathfrak{M}_1, x \models \varrho$ implies $\mathfrak{M}_2, h(x) \models \varrho$, for any $x \in W_1$ and sp-formula ϱ.

4.2.1. *Kripke models from sp-formulas.* We say that a frame $\mathfrak{F} = (W, R^{\mathfrak{F}})_{R\in\mathcal{R}}$ is *tree-shaped* (or simply a *tree*) *with root* r if $(W, \bigcup_{R\in\mathcal{R}} R^{\mathfrak{F}})$ is a finite directed tree with root r such that $R_1^{\mathfrak{F}} \cap R_2^{\mathfrak{F}} = \emptyset$ for all $R_1 \neq R_2$. (In particular, $(W, R^{\mathfrak{F}})_{R\in\mathcal{R}}$ is irreflexive and intransitive.)

We use the following notions and notation throughout the paper. Given an sp-formula ϱ, we define by induction a Kripke model

$$\mathfrak{M}_\varrho = (\mathfrak{T}_\varrho, \mathfrak{v}_\varrho) \text{ based on a finite tree } \mathfrak{T}_\varrho = (W_\varrho, R_\varrho)_{R \in \mathcal{R}} \text{ with root } r_\varrho.$$

For $\varrho = \top$, $\mathfrak{T}_\varrho$ consists of a single irreflexive point r_ϱ with $\mathfrak{v}_\varrho(p) = \emptyset$ for all variables p. For $\varrho = p$, $\mathfrak{T}_\varrho$ consists of a single irreflexive point r_ϱ, $\mathfrak{v}_\varrho(p) = \{r_\varrho\}$, and $\mathfrak{v}_\varrho(q) = \emptyset$ for $q \neq p$. For $\varrho = \varrho_1 \wedge \varrho_2$, we first construct disjoint $\mathfrak{M}_{\varrho_1}$ and $\mathfrak{M}_{\varrho_2}$, and then merge their roots r_{ϱ_1} and r_{ϱ_2} into r_ϱ such that $r_\varrho \in \mathfrak{v}_\varrho(q)$ iff $r_{\varrho_i} \in \mathfrak{v}_{\varrho_i}(q)$, for some $i = 1, 2$. Finally, for $\varrho = \Diamond_R \varrho'$, we add a fresh point r_ϱ to $W_{\varrho'}$, and set $R_\varrho = R_{\varrho'} \cup \{(r_\varrho, r_{\varrho'})\}$ and $\mathfrak{v}_\varrho(p) = \mathfrak{v}_{\varrho'}(p)$ for all variables p. We refer to $\mathfrak{M}_\varrho$ as the *ϱ-tree model*. Note that $\mathfrak{M}_\varrho$ and ϱ are of the same size as the points in W_ϱ are in one-to-one correspondence with the subformulas of ϱ.

Proposition 7. *For any sp-formula ϱ, Kripke model $\mathfrak{M}$ and point w in $\mathfrak{M}$, we have $\mathfrak{M}, w \models \varrho$ iff there is a homomorphism $h \colon \mathfrak{M}_\varrho \to \mathfrak{M}$ with $h(r_\varrho) = w$.*

Proof. By a straightforward induction on the construction of ϱ. $\dashv$

The connection between the validity of sp-implications and homomorphisms between models proved below was first observed in [6].

Corollary 8. (i) *For any sp-implication $\boldsymbol{\iota} = (\sigma \to \tau)$, Kripke model $\mathfrak{M}$ and point w in $\mathfrak{M}$, the following conditions are equivalent*:

- *$\mathfrak{M}, w \models \boldsymbol{\iota}$;*
- *for every homomorphism $h_\sigma \colon \mathfrak{M}_\sigma \to \mathfrak{M}$ with $h_\sigma(r_\sigma) = w$, there is a homomorphism $h_\tau \colon \mathfrak{M}_\tau \to \mathfrak{M}$ with $h_\tau(r_\tau) = w$.*

(ii) *For any sp-formulas σ and τ, we have $\models_{\mathsf{Kr}} \sigma \to \tau$ iff $\mathfrak{M}_\sigma, r_\sigma \models \tau$.*

Proof. Claim (i) is an immediate consequence of Proposition 7.

(ii) $(\Rightarrow)$ As the identity map on $\mathfrak{M}_\sigma$ is a homomorphism, $\mathfrak{M}_\sigma, r_\sigma \models \sigma$ by Proposition 7, and so $\mathfrak{M}_\sigma, r_\sigma \models \tau$ by the assumption.

$(\Leftarrow)$ Suppose $\mathfrak{M}, w \models \sigma$, for some Kripke model $\mathfrak{M}$ based on a frame $\mathfrak{F}$. By Proposition 7, there is a homomorphism $h \colon \mathfrak{M}_\sigma \to \mathfrak{M}$ with $h(r_\sigma) = w$. Thus, $\mathfrak{M}, w \models \tau$ follows from $\mathfrak{M}_\sigma, r_\sigma \models \tau$. $\dashv$

4.2.2. *Sp-formulas from Kripke models.* Suppose $\mathfrak{N} = (\mathfrak{F}, \mathfrak{v})$ is a Kripke model such that $\mathfrak{v}(p) \neq \emptyset$ for finitely many variables p only, and $\mathfrak{F} = (W, R^{\mathfrak{F}})_{R \in \mathcal{R}}$ is a finite frame with root r that contains no directed cycles. We inductively associate with $\mathfrak{N}$ an sp-formula $\mathsf{for}(\mathfrak{N}) = \mathsf{for}_r^{\mathfrak{N}}$ by setting, for every $w \in W$,

$$\mathsf{for}_w^{\mathfrak{N}} = \bigwedge_{w \in \mathfrak{v}(p)} p \;\wedge \bigwedge_{(w,v) \in R^{\mathfrak{F}}} \Diamond_R \, \mathsf{for}_v^{\mathfrak{N}}.$$

Clearly, $\mathfrak{N}, w \models \mathsf{for}_w^{\mathfrak{N}}$. Observe that if $\mathfrak{F}$ is a directed tree then $\mathsf{for}_w^{\mathfrak{N}}$ is the unique (modulo SLO-axioms (2)–(4)) sp-formula ϱ such that the ϱ-tree model $\mathfrak{M}_\varrho$ is the submodel of $\mathfrak{N}$ generated by w. Thus, in this case $\mathfrak{M}_{\mathsf{for}(\mathfrak{N})}$ is the same as $\mathfrak{N}$. In particular, $\mathsf{for}(\mathfrak{M}_\sigma) = \sigma$, for any sp-formula σ. In general, the $\mathsf{for}(\mathfrak{N})$-tree model $\mathfrak{M}_{\mathsf{for}(\mathfrak{N})}$ is what is known in modal logic as the *r-unravelling of* $\mathfrak{N}$, and so:

Proposition 9. *For every sp-formula τ, $\mathfrak{N}, r \models \tau$ iff $\mathfrak{M}_{\mathsf{for}(\mathfrak{N})}, r \models \tau$.*

We also note the following important fact:

PROPOSITION 10. *If* $h\colon \mathfrak{N}_1 \to \mathfrak{N}_2$ *is a homomorphism, then, for every* w *in* $\mathfrak{N}_1$*, we have* $\vdash_{\mathsf{SLO}} \mathsf{for}^{\mathfrak{N}_2}_{h(w)} \to \mathsf{for}^{\mathfrak{N}_1}_{w}$.

4.2.3. *Syntactic proxies.* The above observations give another completeness proof for the spi-logic $\mathsf{SPi} + \emptyset$ (cf. Theorem 4). Indeed, suppose $\models_{\mathsf{Kr}} \sigma \to \tau$. Then, by Corollary 8 (ii), we have $\mathfrak{M}_\sigma, r_\sigma \models \tau$, and so by Proposition 7, there is a homomorphism $h\colon \mathfrak{M}_\tau \to \mathfrak{M}_\sigma$ with $h(r_\tau) = r_\sigma$. Thus, $\vdash_{\mathsf{SLO}} \sigma \to \tau$ follows by Proposition 10, and so $\models_{\mathsf{SLO}} \sigma \to \tau$ by (7).

This proof is a special case of the following general method of establishing completeness of spi-logics, which we call the *method of syntactic proxies*. In order to prove that an spi-logic $\mathsf{SPi} + \Sigma$ is complete (without knowing whether it is complex or not), we do the following, for any given sp-implication $\sigma \to \tau$:

(i) transform one of the sp-formulas σ or τ into some $\Sigma \vdash_{\mathsf{SLO}}$-equivalent normal form resulting in an sp-implication $\alpha \to \beta$, called a Σ-*proxy* for $\sigma \to \tau$;

(ii) show that $\Sigma \models_{\mathsf{Kr}} \alpha \to \beta$ is reducible to $\Sigma^- \models_{\mathsf{Kr}} \alpha \to \beta$, for some subset Σ^- of Σ such that $\mathsf{SPi} + \Sigma^-$ is complete and has the finite frame property.

The concrete Σ-normal form used in this method depends on Σ and reflects the structure of its frames. Say, for $\Sigma = \{\Diamond_R\, p \to \Diamond_S\, p\}$ that defines the property $\Phi = \forall x, y\,(R(x,y) \to S(x,y))$, we transform σ into a $\Sigma \vdash_{\mathsf{SLO}}$-equivalent sp-formula describing the Φ-closure of the finite σ-tree model $\mathfrak{M}_\sigma$, and take $\Sigma^- = \emptyset$ (see Theorem 20). For Σ_{lin} defining linear quasiorders, we transform τ into a conjunction of sp-formulas, each of which describes a linearly ordered full branch of the finite τ-tree model $\mathfrak{M}_\tau$, and take $\Sigma^- = \Sigma_{qo}$ (see Theorem 48).

We use the method of syntactic proxies to obtain a number of completeness results: Theorem 20, which is a general completeness criterion (where we do not know whether all the covered spi-logics are complex), and Theorems 40, 41 and 48 (where the spi-logics in question are *not* complex).

In the next three sections, we apply the tools and techniques developed above to investigate completeness properties of spi-logics, systematising our results according to the form of the first-order correspondents of their axioms.

§5. Completeness of spi-logics with universal Horn correspondents.

We begin by recalling that, by the correspondence part of Sahlqvist's theorem [63, 13], a first-order correspondent $\Psi_{\boldsymbol{\iota}}$ of any sp-implication $\boldsymbol{\iota} = (\sigma \to \tau)$ can be constructed as follows, using the tree models $\mathfrak{M}_\sigma$ and $\mathfrak{M}_\tau$ from §4.2.1. Suppose $W_\sigma = \{v_0, v_1, \dots, v_{n_\sigma}\}$ with $v_0 = r_\sigma$, and $W_\tau = \{u_0, u_1, \dots, u_{n_\tau}\}$ with $u_0 = r_\tau$. With each point w in $W_\sigma \cup W_\tau$, we associate a variable $\hat{w}$, and set

$$(18)\quad \Psi'_{\boldsymbol{\iota}}(\hat{v}_0) \;=\; \forall \hat{v}_1, \dots, \hat{v}_{n_\sigma} \Big(\bigwedge_{\substack{i,j \le n_\sigma,\, R \in \mathcal{R} \\ (v_i, v_j) \in R_\sigma}} R(\hat{v}_i, \hat{v}_j) \to \exists \hat{u}_0, \dots, \hat{u}_{n_\tau} \big((\hat{v}_0 = \hat{u}_0) \wedge \bigwedge_{\substack{i,j \le n_\tau,\, R \in \mathcal{R} \\ (u_i, u_j) \in R_\tau}} R(\hat{u}_i, \hat{u}_j) \wedge \bigwedge_{\substack{i \le n_\tau,\, p \in \mathsf{var} \\ u_i \in \mathfrak{v}_\tau(p)}} \bigvee_{\substack{j \le n_\tau \\ v_j \in \mathfrak{v}_\sigma(p)}} (\hat{u}_i = \hat{v}_j) \big) \Big).$$

Then (as actually follows from Corollary 8 (i)), for any frame $\mathfrak{F}$ and any point w in it, $\mathfrak{F}, w \models \boldsymbol{\iota}$ iff $\mathfrak{F} \models \Psi'_{\boldsymbol{\iota}}(\hat{v}_0)[w]$. The formula $\Psi'_{\boldsymbol{\iota}}(\hat{v}_0)$ with one free variable $\hat{v}_0$ is called a *local correspondent of* $\boldsymbol{\iota}$. The sentence $\Psi_{\boldsymbol{\iota}} = \forall \hat{v}_0\, \Psi'_{\boldsymbol{\iota}}(\hat{v}_0)$ is then a (*global*) *correspondent of* $\boldsymbol{\iota}$, that is, for every frame $\mathfrak{F}$,

$$\mathfrak{F} \models \boldsymbol{\iota} \quad \Longleftrightarrow \quad \mathfrak{F} \models \Psi_{\boldsymbol{\iota}}. \tag{19}$$

The left-hand side of the implication in $\Psi_{\boldsymbol{\iota}}$ is just the diagram of the tree-shaped frame $\mathfrak{T}_\sigma$ constructed from the atoms $R(\hat{v}_i, \hat{v}_j)$ with $(v_i, v_j) \in R_\sigma$. The right-hand side has a more complex structure that involves equality, disjunction and existential quantifiers. In some cases, $\Psi_{\boldsymbol{\iota}}$ is equivalent to a first-order sentence without some of these. For example, reflexivity, transitivity or symmetry can clearly be defined without using any of $=$, $\vee$ and $\exists$ on the right-hand side. On the other hand, $=$ is required to define functionality, $\vee$ is needed for linearity, and $\exists$ for density. Note that if $\Psi_{\boldsymbol{\iota}}$ is equivalent to a universal sentence, then every subframe of a frame in $\mathsf{Kr}_{\{\boldsymbol{\iota}\}}$ is also in $\mathsf{Kr}_{\{\boldsymbol{\iota}\}}$. We call an spi-logic L a *subframe logic* if Kr_L is closed under taking subframes.

THEOREM 11. *Every subframe spi-logic L has the polynomial finite frame property, and is decidable in* coNP *if complete and finitely axiomatisable.*

PROOF. Decidability in coNP follows from completeness and finite axiomatisability, using the polynomial finite frame property. To show it, suppose $\boldsymbol{\iota} \notin L$ and $\boldsymbol{\iota} = (\sigma \to \tau)$. Then there is a Kripke model $\mathfrak{M} \not\models \boldsymbol{\iota}$ based on some $\mathfrak{F} \in \mathsf{Kr}_L$, that is, $\mathfrak{M}, w \models \sigma$ and $\mathfrak{M}, w \not\models \tau$, for some point w. By Proposition 7, there is a homomorphism $h\colon \mathfrak{M}_\sigma \to \mathfrak{M}$ with $h(r_\sigma) = w$. Take the restrictions $\mathfrak{F}'$ and $\mathfrak{M}'$ of, respectively, $\mathfrak{F}$ and $\mathfrak{M}$ to $\{h(w) \mid w \in W_\sigma\}$. Then $\mathfrak{M}' \not\models \boldsymbol{\iota}$ and $\mathfrak{F}' \in \mathsf{Kr}_L$ is a subframe of $\mathfrak{T}_\sigma$, and so it is of polynomial size in $\boldsymbol{\iota}$. $\dashv$

5.1. Equality-free universal Horn correspondents. By a *profile* we mean a quadruple $\pi = (\mathfrak{G}, S, u, v)$, where $\mathfrak{G} = (\Delta, R^{\mathfrak{G}})_{R \in \mathcal{R}}$ is a finite rooted frame with $u, v \in \Delta$, $S \in \mathcal{R}$ and $(u, v) \notin S^{\mathfrak{G}}$. Let $\Delta = \{x_0, \dots, x_n\}$. The profile π represents the universal Horn sentence

$$\Phi_\pi \;=\; \forall \hat{x}_0, \dots, \hat{x}_n \,\big(\bigwedge_{\substack{i,j \le n,\, R \in \mathcal{R} \\ (x_i, x_j) \in R^{\mathfrak{G}}}} R(\hat{x}_i, \hat{x}_j) \to S(\hat{u}, \hat{v}) \big).$$

We call $\boldsymbol{\iota}$ a *Horn-implication* if its correspondent $\Psi_{\boldsymbol{\iota}}$ is equivalent to Φ_π for some profile π, in which case we say that π *is a profile of* $\boldsymbol{\iota}$ or $\boldsymbol{\iota}$ *has profile* π. Since $(u, v) \notin S^{\mathfrak{G}}$, we have $\mathfrak{G} \not\models \Phi_\pi$, and so $\mathfrak{G} \not\models \Psi_{\boldsymbol{\iota}}$. Thus,

$$\text{if } \pi = (\mathfrak{G}, S, u, v) \text{ is a profile of } \boldsymbol{\iota}, \text{ then } \mathfrak{G} \not\models \boldsymbol{\iota}. \tag{20}$$

Given a set Π of profiles and a frame $\mathfrak{F} = (W, R^{\mathfrak{F}})_{R \in \mathcal{R}}$, we denote by $\Pi(\mathfrak{F})$ the Π-*closure of* $\mathfrak{F}$, that is, the smallest frame $\mathfrak{H}$ extending $\mathfrak{F}$ such that $\mathfrak{H} \models \Phi_\pi$, for $\pi \in \Pi$. If $\Pi = \{\pi\}$, we write $\pi(\mathfrak{F})$ instead of $\Pi(\mathfrak{F})$. Thus, $\pi(\mathfrak{F})$ contains the same points as $\mathfrak{F}$ but possibly more S-arrows between them. For a Kripke model $\mathfrak{M} = (\mathfrak{F}, \mathfrak{v})$, we set $\Pi(\mathfrak{M}) = \big(\Pi(\mathfrak{F}), \mathfrak{v}\big)$. Clearly, if both Π and $\mathfrak{F} = (W, R^{\mathfrak{F}})_{R \in \mathcal{R}}$ are finite, we can construct $\Pi(\mathfrak{F})$ step-by-step by defining a finite sequence

$$\mathfrak{F} = \mathfrak{F}^0, \dots, \mathfrak{F}^i = (W, R^{\mathfrak{F}^i})_{R \in \mathcal{R}}, \dots, \mathfrak{F}^n = \Pi(\mathfrak{F}) \tag{21}$$

of frames such that $n \leq |\mathcal{R}| \cdot |W|^2$ and, for every $i < n$, there exist a profile $\pi^i = (\mathfrak{G}^i, S^i, u^i, v^i)$ in Π and a homomorphism $h^i \colon \mathfrak{G}^i \to \mathfrak{F}^i$ with

$$R^{\mathfrak{F}^{i+1}} = \begin{cases} R^{\mathfrak{F}^i} \cup \bigl\{\bigl(h^i(u^i), h^i(v^i)\bigr)\bigr\}, & \text{if } R = S^i, \\ R^{\mathfrak{F}^i}, & \text{otherwise.} \end{cases} \tag{22}$$

To put it another way, $\Pi(\mathfrak{F})$ is the result of applying the datalog program with rules $\{\Phi_\pi \mid \pi \in \Pi\}$ to the input database $\{R^{\mathfrak{F}} \mid R \in \mathcal{R}\}$, which can be done in polynomial time in $\mathfrak{F}$ for a fixed finite Π [24]. In general, using a similar step-by-step construction for successor ordinals and taking the union for limits, one can show that, for any frames $\mathfrak{F}, \mathfrak{F}'$ and set Π of profiles,

(23) any homomorphism $f \colon \mathfrak{F} \to \mathfrak{F}'$ is a $\Pi(\mathfrak{F}) \to \Pi(\mathfrak{F}')$ homomorphism.

We have the following generalisation of Corollary 8 (ii):

Proposition 12. *Let Σ be a set of Horn-implications and $\Pi_\Sigma = \{\pi_{\boldsymbol{\iota}} \mid \boldsymbol{\iota} \in \Sigma\}$ their profiles. Then $\Sigma \models_{\mathsf{Kr}} \sigma \to \tau$ iff $\Pi_\Sigma(\mathfrak{M}_\sigma), r_\sigma \models \tau$, for any sp-formulas σ and τ.*

Proof. ($\Rightarrow$) As $\Pi_\Sigma(\mathfrak{M}_\sigma)$ extends the σ-tree model $\mathfrak{M}_\sigma$, the identity map is an $\mathfrak{M}_\sigma \to \Pi_\Sigma(\mathfrak{M}_\sigma)$ homomorphism, and so $\Pi_\Sigma(\mathfrak{M}_\sigma), r_\sigma \models \sigma$ by Proposition 7. As $\Pi_\Sigma(\mathfrak{T}_\sigma) \models \Phi_{\pi_e}$ for every $\pi_{\boldsymbol{\iota}} \in \Pi_\Sigma$, we have $\Pi_\Sigma(\mathfrak{T}_\sigma) \models \Psi_{\boldsymbol{\iota}}$ for every $\boldsymbol{\iota} \in \Sigma$, and so $\Pi_\Sigma(\mathfrak{T}_\sigma) \models \Sigma$. Therefore, $\Pi_\Sigma(\mathfrak{T}_\sigma) \models \sigma \to \tau$, and so $\Pi_\Sigma(\mathfrak{M}_\sigma), r_\sigma \models \tau$.

($\Leftarrow$) Suppose $\mathfrak{M}, w \models \sigma$ for some Kripke model $\mathfrak{M}$ based on a frame $\mathfrak{F} \in \mathsf{Kr}_\Sigma$. By Proposition 7, there is a homomorphism $h \colon \mathfrak{M}_\sigma \to \mathfrak{M}$ with $h(r_\sigma) = w$. By (23), h is a homomorphism from $\Pi(\mathfrak{M}_\sigma)$ to $\Pi(\mathfrak{M})$. As $\mathfrak{F} \models \Sigma$, we have $\mathfrak{F} \models \Phi_{\pi_{\boldsymbol{\iota}}}$ for any $\pi_{\boldsymbol{\iota}} \in \Pi_\Sigma$. Thus, $\Pi_\Sigma(\mathfrak{M}) = \mathfrak{M}$, and so $\mathfrak{M}, w \models \tau$ follows from $\Pi_\Sigma(\mathfrak{M}_\sigma), r_\sigma \models \tau$, as required. ⊣

As the Kripke model $\Pi_\Sigma(\mathfrak{M}_\sigma)$ has $|W_\sigma|$-many points and can be constructed in polynomial time in σ, we obtain the following consequence of Proposition 12:

Theorem 13. *For any finite set Σ of Horn-implications, $\mathsf{SPi} + \Sigma$ has the polynomial finite frame property, and is decidable in* PTime *if complete.*

Note that full Boolean normal multi-modal logics axiomatisable by Horn-implications can be very complex. For example, it is shown in [52] that $\mathsf{K} \oplus \Sigma$ is undecidable for

$$\Sigma = \{\Diamond_R \Diamond_P \Diamond_R\, p \to \Diamond_P\, p,\ \Diamond_Q \Diamond_R\, p \to \Diamond_Q\, p,\ \Diamond_Q \Diamond_P\, p \to \Diamond_P\, p\},$$

On the other hand, by Corollary 16 below, the spi-logic $\mathsf{SPi} + \Sigma$ is complete, and so decidable in PTime by Theorem 13. For more decidability and complexity results for modal logics of Horn definable classes of frames, the reader is referred to [45, 59].

In the remainder of this section, we provide a few general sufficient conditions for completeness of spi-logics axiomatisable by Horn-implications, and also give a number of counterexamples illustrating their boundaries.

We say that $\pi = (\mathfrak{G}, S, u, v)$ is a *tree-profile* if $\mathfrak{G}$ is a tree with root $r_{\mathfrak{G}}$.

Proposition 14. *Suppose that a Horn-implication $\boldsymbol{\iota} = (\sigma \to \tau)$ has a tree-profile $(\mathfrak{G}, S, u, v)$. Then the following hold:*

(i) there exist a homomorphism $f \colon \mathfrak{T}_\sigma \to \mathfrak{G}$ and a homomorphism $g \colon \mathfrak{G} \to \mathfrak{T}_\sigma$;

(*ii*) *for any homomorphism* $h\colon \mathfrak{T}_\sigma \to \mathfrak{G}$, *we have* $h(r_\sigma) = r_{\mathfrak{G}}$.

PROOF. (*i*) By (20) $\mathfrak{G} \not\models \boldsymbol{\iota}$, and so there is a homomorphism $f\colon \mathfrak{T}_\sigma \to \mathfrak{G}$. Since $\not\models_{\mathsf{Kr}} \boldsymbol{\iota}$, by Corollary 8 (*ii*) we obtain $\mathfrak{M}_\sigma, r_\sigma \not\models \tau$, from which $\mathfrak{T}_\sigma \not\models \boldsymbol{\iota}$. Therefore, $\mathfrak{T}_\sigma \not\models \Phi_\pi$, and so there is a homomorphism $g\colon \mathfrak{G} \to \mathfrak{T}_\sigma$.

(*ii*) Suppose $h\colon \mathfrak{T}_\sigma \to \mathfrak{G}$ is a homomorphism. Then the composition of g and h is a homomorphism from the finite tree $\mathfrak{G}$ to itself, which gives $h\big(g(r_{\mathfrak{G}})\big) = r_{\mathfrak{G}}$, and so $g(r_{\mathfrak{G}}) = r_\sigma$ must hold as well. ⊣

A profile $\pi = (\mathfrak{G}, S, u, v)$ is *minimal* if there is no profile $\pi' = (\mathfrak{G}', S', u', v')$ such that $|\mathfrak{G}'| < |\mathfrak{G}|$ and Φ_π is equivalent to $\Phi_{\pi'}$. As shown in [50], for any minimal profile π, the class of frames validating Φ_π is modally definable iff π is a tree-profile. (Thus, every Horn-implication has a correspondent Φ_π given by a minimal tree-profile π.) Moreover, any such modally definable class is in fact definable by a single sp-implication $\boldsymbol{\iota}_\pi$ constructed in the following way.

Suppose $y_0 R_1^{\mathfrak{G}} y_1 \dots y_{\ell-1} R_\ell^{\mathfrak{G}} y_\ell$ is the unique path in the tree-shaped frame $\mathfrak{G} = (\Delta, R^{\mathfrak{G}})_{R \in \mathcal{R}}$ from the root y_0 to $y_\ell = u$, for some $\ell < \omega$. We introduce a propositional variable p_x for each $x \in \{y_1, \dots, y_\ell, v\}$. Let $\Delta = \{x_0, \dots, x_n\}$ be such that $x_0 = y_0$ is the root of $\mathfrak{G}$, and $(x_i, x_j) \in R^{\mathfrak{G}}$ implies $i < j$, for all $i, j \le n$ and $R \in \mathcal{R}$. By induction on i from n to 0, we set

$$\sigma_i = \begin{cases} p_x \wedge \bigwedge\limits_{(x_i,x_j) \in R^{\mathfrak{G}}} \Diamond_R\, \sigma_j, & \text{if } x_i = x \text{ for some } x \in \{y_1, \dots, y_\ell, v\}, \\ \top \wedge \bigwedge\limits_{(x_i,x_j) \in R^{\mathfrak{G}}} \Diamond_R\, \sigma_j, & \text{otherwise} \end{cases} \tag{24}$$

and

$$\boldsymbol{\iota}_\pi = \Big(\sigma_0 \to \Diamond_{R_1}\big(p_{y_1} \wedge \Diamond_{R_2}\big(p_{y_2} \wedge \dots \wedge \Diamond_{R_\ell}\big(p_u \wedge \Diamond_S\, p_v\big)\dots\big)\big)\Big). \tag{25}$$

It is readily checked that $\boldsymbol{\iota}_\pi$ is a Horn-implication and π is a profile of $\boldsymbol{\iota}_\pi$.

5.1.1. *Horn-implications with rooted tree-profiles.* We say that a tree-profile $\pi = (\mathfrak{G}, S, u, v)$ is *rooted* if u is the root of $\mathfrak{G}$, in which case

$$\boldsymbol{\iota}_\pi \;=\; (\sigma_0 \to \Diamond_S\, p_v),$$

and the only variable that occurs in σ_0 is p_v. A few examples of tree-profiles π with their Φ_π and $\boldsymbol{\iota}_\pi$ are given in Table 3, where the first two tree-profiles (for reflexivity and transitivity) are rooted, and the last two (for symmetry and Euclideanness) are non-rooted.

THEOREM 15. *Any spi-logic axiomatised by sp-implications* $\boldsymbol{\iota}_\pi$, *for some rooted tree-profiles* π, *is complex, and so complete.*

A generalisation of this theorem (Theorem 35) will be proved in §6. Note that as a consequence we obtain the following:

COROLLARY 16 ([71]). *Any spi-logic axiomatised by sp-implications of the form* $\Diamond_{R_1} \dots \Diamond_{R_n} p \to \Diamond_{R_0} p$, *for* $n \ge 0$, *is complex, and so complete. In particular,* $\mathsf{SPi} + \{\boldsymbol{\iota}_{\mathit{refl}}\}$, $\mathsf{SPi} + \{\boldsymbol{\iota}_{\mathit{trans}}\}$, *and* SPi_{qo} *are all complex and complete.*

In general, there can be different sp-implications $\boldsymbol{\iota}$ with the same rooted tree-profile π. Since for each such $\boldsymbol{\iota}$, $\Psi_{\boldsymbol{\iota}}$ is equivalent to Φ_π, $\boldsymbol{\iota}$ and $\boldsymbol{\iota}_\pi$ are valid in the same frames. However, we do not necessarily have $\mathsf{SLO}_{\{e\}} = \mathsf{SLO}_{\{\boldsymbol{\iota}_\pi\}}$, and so

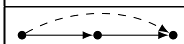
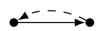
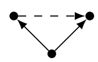
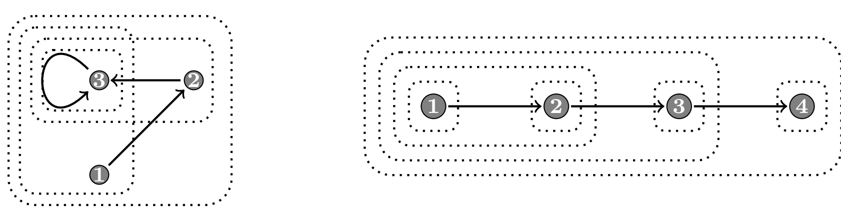
profile π	Φ_π	$\boldsymbol{\iota}_\pi$
	$\forall x\, R(x,x)$	$\boldsymbol{\iota}_{\mathit{refl}}:\ p \to \Diamond p$
	$\forall x,y,z\, \big(R(x,y) \land R(y,z) \to R(x,z)\big)$	$\boldsymbol{\iota}_{\mathit{trans}}:\ \Diamond\Diamond p \to \Diamond p$
	$\forall x,y\, \big(R(x,y) \to R(y,x)\big)$	$\boldsymbol{\iota}_{\mathit{sym}}:$ $q \land \Diamond p \to \Diamond(p \land \Diamond q)$
	$\forall x,y,z\, \big(R(x,y) \land R(x,z) \to R(y,z)\big)$	$\boldsymbol{\iota}_{\mathit{eucl}}:$ $\Diamond p \land \Diamond q \to \Diamond(p \land \Diamond q)$

TABLE 3. Examples of rooted and non-rooted tree-profiles.

Theorem 15 cannot be generalised to all such sp-implications, as shown by the following examples.

EXAMPLE 17. Consider first the rooted tree-profile π for reflexivity in Table 3. It is not hard to see that the sp-implication $\boldsymbol{\iota} = \big(p \to \Diamond\Diamond(p \land \Diamond p)\big)$ also has Φ_π as its correspondent, and so $\boldsymbol{\iota}_\pi = \boldsymbol{\iota}_{\mathit{refl}}$ is valid in exactly the same frames as $\boldsymbol{\iota}$. On the other hand, $\{\boldsymbol{\iota}\} \not\models_{\mathsf{SLO}} \boldsymbol{\iota}_\pi$ because the SLO in Fig. 2 (a) validates $\boldsymbol{\iota}$ but refutes $\boldsymbol{\iota}_\pi$ when p is $\{1,3\}$. Therefore, $\mathsf{SPi} + \{\boldsymbol{\iota}\}$ is not complete.

EXAMPLE 18. Let $\pi = (\mathfrak{G}, R, v_1, v_4)$, where $\mathfrak{G}$ is an R-chain of v_1, v_2, v_3, v_4. It is not hard to check that $\Phi_\pi = \forall \boldsymbol{x}\, \big(R(x_1,x_2) \land R(x_2,x_3) \land R(x_3,x_4) \to R(x_1,x_4)\big)$ is a correspondent of the sp-implication $\boldsymbol{\iota}' = (\Diamond\Diamond p \land \Diamond\Diamond\Diamond p \to \Diamond p)$. The SLO in Fig. 2 (b) validates $\boldsymbol{\iota}'$ but refutes $\boldsymbol{\iota}_\pi = (\Diamond\Diamond\Diamond p \to \Diamond p)$ when p is $\{4\}$. Therefore, $\{\boldsymbol{\iota}'\} \not\models_{\mathsf{SLO}} \boldsymbol{\iota}_\pi$, and so $\mathsf{SPi} + \{\boldsymbol{\iota}'\}$ is not complete.



(a) (b)

FIGURE 2. The SLOs of Examples 17 and 18.

We say that a rooted tree-profile $\pi = (\mathfrak{G}, S, u, v)$ is *leapfrog* if $(u, w) \notin S^{\mathfrak{G}}$ for any w in $\mathfrak{G}$; and we refer to a Horn-implication of the form $\varrho \to \Diamond_S\, p$ having a leapfrog profile as a *leapfrog implication.*

THEOREM 19. *Any spi-logic axiomatised by leapfrog implications is complex, and so complete.*

PROOF. Suppose $\boldsymbol{\iota} = (\varrho \to \Diamond_S\, p)$ is a Horn-implication with a leapfrog profile $\pi = (\mathfrak{G}, S, u, v)$. Recall the finite tree $\mathfrak{T}_\varrho = (W_\varrho, R_\varrho)_{R \in \mathcal{R}}$ with root r_ϱ from

§4.2.1. By Proposition 14, we obtain that

$$\text{there is no } z \text{ with } (r_\varrho, z) \in S_\varrho. \tag{26}$$

CLAIM 19.1. (*i*) *For every* $y \in \mathfrak{v}_\varrho(p)$, *there is a homomorphism* $h^y \colon \mathfrak{T}_\varrho \to \mathfrak{G}$ *with* $h^y(r_\varrho) = u$ *and* $h^y(y) = v$.

(*ii*) *There is a homomorphism* $h \colon \mathfrak{T}_\varrho \to \mathfrak{G}$ *such that* $h(r_\varrho) = u$ *and* $h(y) = v$, *for all* $y \in \mathfrak{v}_\varrho(p)$.

PROOF. (*i*) Fix some $y \in \mathfrak{v}_\varrho(p)$ and consider the rooted tree-profile $\pi_{\varrho,y} = (\mathfrak{T}_\varrho, S, r_\varrho, y)$. With each point x in W_ϱ we associate a variable $\hat{x}$. As

$$\Phi_{\pi_{\varrho,y}} = \forall \hat{\boldsymbol{x}} \big(\bigwedge_{\substack{x,x' \in W_\varrho,\, R \in \mathcal{R}, \\ (x,x') \in R_\varrho}} R(\hat{x}, \hat{x}') \to S(\hat{r}_\varrho, \hat{y}) \big), \quad \text{and}$$

$$\Phi_\pi \leftrightarrow \Psi_{\boldsymbol{\iota}} \leftrightarrow \forall \hat{\boldsymbol{x}} \big(\bigwedge_{\substack{x,x' \in W_\varrho,\, R \in \mathcal{R}, \\ (x,x') \in R_\varrho}} R(\hat{x}, \hat{x}') \to \bigvee_{y \in \mathfrak{v}_\varrho(p)} S(\hat{r}_\varrho, \hat{y}) \big),$$

$\Phi_{\pi_{\varrho,y}}$ implies Φ_π. Take the $\pi_{\varrho,y}$-closure $\pi_{\varrho,y}(\mathfrak{G})$ of $\mathfrak{G}$. As $\pi_{\varrho,y}(\mathfrak{G}) \models \Phi_{\pi_{\varrho,y}}$, we have $\pi_{\varrho,y}(\mathfrak{G}) \models \Phi_\pi$. As the identity map is a homomorphism from $\mathfrak{G}$ to $\pi_{\varrho,y}(\mathfrak{G})$,

$$(u, v) \in R^{\pi_{\varrho,y}(\mathfrak{G})}. \tag{27}$$

Next, consider the step-by-step construction (21)–(22) of $\pi_{\varrho,y}(\mathfrak{G})$. We show by induction that, for every $i < n$, (*a*) the homomorphism $h^i \colon \mathfrak{T}_\varrho \to \mathfrak{F}^i$ used to obtain $\mathfrak{F}^{i+1}$ from $\mathfrak{F}^i$ is in fact a $\mathfrak{T}_\varrho \to \mathfrak{G}$ homomorphism, and so, by Proposition 14 (*ii*), (*b*) the new pair in $S^{\mathfrak{F}^{i+1}}$ is $\big(u, h^i(y)\big)$. Indeed, for $i = 0$ this follows from $\mathfrak{F}^0 = \mathfrak{G}$. Now suppose inductively that (*a*) and (*b*) hold for all $j \leq i$, and take the homomorphism $h^{i+1} \colon \mathfrak{T}_\varrho \to \mathfrak{F}^{i+1}$. Since by IH all the S-pairs in $\mathfrak{F}^{i+1}$ that are not in $\mathfrak{G}$ are of the form (u, z), for some z, (26) implies that h^{i+1} is a $\mathfrak{T}_\varrho \to \mathfrak{G}$ homomorphism, proving (*a*). Now by (27) and (*a*), there is $i < n$ such that $h^i(r_\varrho) = u$ and $h^i(y) = v$, for the homomorphism $h^i \colon \mathfrak{T}_\varrho \to \mathfrak{G}$, as required.

(*ii*) We define a homomorphism $h \colon \mathfrak{T}_\varrho \to \mathfrak{G}$ as follows. First, define h on the *trunk* of $\mathfrak{T}_\varrho$ comprising the points that lie on the paths from r_ϱ to some $y \in \mathfrak{v}_\varrho(p)$. Namely, for each z on the trunk, we take any y such that z lies on the path from r_ϱ to y and set $h(z) = h^y(z)$ (which is well-defined since $\mathfrak{G}$ is a tree, and so all the y are located at the same distance from r_ϱ). Next, for any d on the trunk, we take the *branch* with *base* d (containing all non-trunk descendants of d), fix some y such that $y \in \mathfrak{v}_\varrho(p)$ and d lies on the path from r_ϱ to y, and set $h(z) = h^y(z)$ for any z on that branch. It is readily seen that h is as required. ⊣

Now, let $\mathfrak{A} = (A, \wedge, \top, \Diamond_R)_{R \in \mathcal{R}}$ be a SLO validating $\boldsymbol{\iota}$. It is shown in §4.1.1 that $\mathfrak{A}$ can be embedded into $\mathfrak{F}^\star$, for the frame $\mathfrak{F} = (A, R^{\mathfrak{F}})_{R \in \mathcal{R}}$ with $R^{\mathfrak{F}}$ given by (14). We show that $\mathfrak{F} \models \Phi_\pi$, and so $\mathfrak{F} \models \Psi_{\boldsymbol{\iota}}$, as required. To begin with, take the tree-shaped frame $\mathfrak{G} = (\Delta, R^{\mathfrak{G}})_{R \in \mathcal{R}}$ and suppose that $\Delta = \{x_1, \dots, x_n\}$ such that $x_1 = u$ is the root, and $(x_i, x_j) \in R^{\mathfrak{G}}$ implies $i < j$. For each $i = 1, \dots, n$, take some $a_{x_i} \in A$ such that $(a_{x_i}, a_{x_j}) \in R^{\mathfrak{F}}$ whenever $(x_i, x_j) \in R^{\mathfrak{G}}$. We need to show that $(a_u, a_v) \in S^{\mathfrak{F}}$, that is, $a_u \leq \Diamond_S\, a_v$. Take the sp-formulas σ_i defined in (24). We prove by induction on $i = n, \dots, 0$ that

$$a_{x_i} \leq \sigma_i[a_v]. \tag{28}$$

Indeed, as x_n is a leaf in $\mathfrak{G}$, σ_n is either $\top$ (if $x_n \neq v$) or p_v (if $x_n = v$), and so in either case (28) holds for a_{x_n}. Now suppose inductively that (28) holds for every j, $i < j \leq n$. We have $a_{x_i} \leq \Diamond_R\, a_{x_j}$ for every x_j with $(x_i, x_j) \in R^{\mathfrak{G}}$. So, by IH and monotonicity, we have

$$a_{x_i} \leq a_{x_i} \wedge \bigwedge_{(x_i,x_j)\in R^{\mathfrak{G}}} \Diamond_R\, \sigma_j[a_v].$$

Since

$$\sigma_i[a_v] = \begin{cases} \top \wedge \displaystyle\bigwedge_{(x_i,x_j)\in R^{\mathfrak{G}}} \Diamond_R\, \sigma_j[a_v], & \text{if } x_i \neq v,\\ a_v \wedge \displaystyle\bigwedge_{(x_i,x_j)\in R^{\mathfrak{G}}} \Diamond_R\, \sigma_j[a_v], & \text{if } x_i = v, \end{cases}$$

(28) follows. In particular, we have $a_u = a_{x_1} \leq \sigma_1[a_v]$.

Now, take the following valuation $\mathfrak{a}$ in $\mathfrak{A}$, for any variable q:

$$\mathfrak{a}(q) = \begin{cases} a_v, & \text{if } q = p,\\ \top, & \text{otherwise,} \end{cases}$$

and take the homomorphism h from Claim 19.1 (*ii*). For any y in $\mathfrak{T}_\varrho$, take the sp-formula $\mathsf{for}_y^{\mathfrak{M}_\varrho}$ defined in §4.2.2. One can readily show by induction that

$$h(y) = x_i \ \text{ implies } \ \sigma_i[a_v] \leq \mathsf{for}_y^{\mathfrak{M}_\varrho}[\mathfrak{a}].$$

Indeed, if y is a leaf in $\mathfrak{T}_\varrho$ and $y \notin \mathfrak{v}_\varrho(p)$, then $\mathsf{for}_y^{\mathfrak{M}_\varrho}[\mathfrak{a}] = \top$. If y is a leaf and $y \in \mathfrak{v}_\varrho(p)$, then $h(y) = v$, and so $\sigma_i[a_v] \leq a_v = \mathsf{for}_y^{\mathfrak{M}_\varrho}[\mathfrak{a}]$. If $y \in \mathfrak{v}_\varrho(p)$ and has ℓ successors $y_0, \dots, y_{\ell-1}$ with $(y, y_j) \in R^j_\varrho$, then by IH and monotonicity, we have

$$\sigma_i[a_v] \leq a_v \wedge \bigwedge_{\substack{x_k = h(y_j)\\ \text{for some } j<\ell}} \Diamond_{R^j}\, \sigma_k[a_v] \leq a_v \wedge \bigwedge_{j<\ell} \Diamond_{R^j}\, \mathsf{for}_{y_j}^{\mathfrak{M}_\varrho}[\mathfrak{a}] = \mathsf{for}_y^{\mathfrak{M}_\varrho}[\mathfrak{a}].$$

The case $y \notin \mathfrak{v}_\varrho(p)$ is similar. In particular, we have $\sigma_1[a_v] \leq \varrho[\mathfrak{a}]$. Finally, as $\mathfrak{A} \models \varrho \to \Diamond_S\, p$, we obtain $\sigma_1[a_v] \leq \Diamond_S\, a_v$, and so $a_u \leq \Diamond_S\, a_v$ by (28). $\dashv$

5.1.2. *Horn-implications with arbitrary tree-profiles.* We consider next Horn-implications with tree-profiles $\pi = (\mathfrak{G}, S, u, v)$ such that u is not necessarily the root of the tree $\mathfrak{G}$. Here again there are both positive and negative results. We begin by proving a general sufficient condition for completeness.

A set Π of tree-profiles is called *stable* if, for any $\pi = (\mathfrak{G}, S, u, v)$ in Π and any tree $\mathfrak{T}$, every homomorphism $h\colon \mathfrak{G} \to \Pi(\mathfrak{T})$ is also a homomorphism from $\mathfrak{G}$ to $\mathfrak{T}$. To illustrate, $\{\pi_1\}$ and $\{\pi_2\}$ in Fig. 3 are stable, while $\{\pi_3\}$ is not (take the 'linear' frame $\mathfrak{T}$ with $S^{\mathfrak{T}} = \{(u_1, u_2)\}$ and $R^{\mathfrak{T}} = \{(u_2, u_3), (u_3, u_4)\}$). We say that a tree-profile $\pi = (\mathfrak{G}, S, u, v)$ is *forward-looking* if $u <_{\mathfrak{G}} v$, where $<_{\mathfrak{G}}$ is the transitive closure of $\bigcup_{R\in\mathcal{R}} R^{\mathfrak{G}}$.

Suppose a tree-profile $\pi = (\mathfrak{G}, S, u, v)$ is forward-looking and $\mathfrak{G} = (\Delta, R^{\mathfrak{G}})_{R\in\mathcal{R}}$. We define an sp-implication $\boldsymbol{\iota}'_\pi$ as follows. For every $x \in \Delta$, we take a propositional variable p_x, and denote by $\mathfrak{v}$ the valuation given by $\mathfrak{v}(p_x) = \{x\}$. Let $\mathfrak{M} = (\mathfrak{G}, \mathfrak{v})$ and $\mathfrak{M}' = (\mathfrak{G}', \mathfrak{v})$, where $\mathfrak{G}' = (\Delta, R^{\mathfrak{G}'})_{R\in\mathcal{R}}$ with $R^{\mathfrak{G}'} = R^{\mathfrak{G}}$, for $R \neq S$, and $S^{\mathfrak{G}'} = S^{\mathfrak{G}} \cup \{(u, v)\}$. Since π is forward-looking, $\mathfrak{G}'$ does not contain directed cycles, and so both sp-formulas $\mathsf{for}(\mathfrak{M})$ and $\mathsf{for}(\mathfrak{M}')$ are defined (see

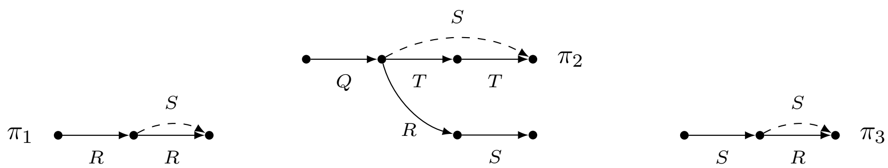


FIGURE 3. Stable and unstable tree-profiles.

§4.2.2), with $\mathsf{for}(\mathfrak{M}')$ obtained by substituting $p_u \wedge \Diamond_S \mathsf{for}_v^{\mathfrak{M}}$ for p_u in $\mathsf{for}(\mathfrak{M})$. We set

$$\boldsymbol{\iota}'_\pi = \big(\mathsf{for}(\mathfrak{M}) \to \mathsf{for}(\mathfrak{M}')\big).$$

It is readily checked that π is a profile of $\boldsymbol{\iota}'_\pi$. The difference between $\boldsymbol{\iota}'_\pi$ and the sp-implication $\boldsymbol{\iota}_\pi$ defined by (25) is that the former contains propositional variables for all points in $\mathfrak{G}$, while the latter only for v and for the points on the path from the root of $\mathfrak{G}$ to u. For example, for the transitivity profile π from Table 3, we have

$$\boldsymbol{\iota}'_\pi = \big(p_1 \wedge \Diamond(p_2 \wedge \Diamond p_3) \to p_1 \wedge \Diamond(p_2 \wedge \Diamond p_3) \wedge \Diamond p_3\big) \quad \text{and} \quad \boldsymbol{\iota}_\pi = \big(\Diamond\Diamond p \to \Diamond p\big).$$

The extra variables make it possible to obtain the following:

THEOREM 20. *For any stable set* Π *of forward-looking tree-profiles, the spi-logic* $\mathsf{SPi} + \Sigma'_\Pi$, *for* $\Sigma'_\Pi = \{\boldsymbol{\iota}'_\pi \mid \pi \in \Pi\}$, *is complete.*

PROOF. The proof uses the syntactic proxies method from §4.2. Given an sp-formula σ, we take the Π-closure $\Pi(\mathfrak{M}_\sigma)$ of its tree-model $\mathfrak{M}_\sigma$. As every $\pi \in \Pi$ is forward-looking, $\Pi(\mathfrak{T}_\sigma)$ does not contain directed cycles, and so the sp-formula $\mathsf{for}\big(\Pi(\mathfrak{M}_\sigma)\big)$ is defined in §4.2.2. We show that $\varrho_\sigma = \mathsf{for}\big(\Pi(\mathfrak{M}_\sigma)\big)$ has the following properties:

(i) for any sp-formula τ, if $\Sigma'_\Pi \models_{\mathsf{Kr}} \varrho_\sigma \to \tau$ then $\models_{\mathsf{Kr}} \varrho_\sigma \to \tau$,
(ii) $\Sigma'_\Pi \vdash_{\mathsf{SLO}} \varrho_\sigma \to \sigma$ and $\Sigma'_\Pi \vdash_{\mathsf{SLO}} \sigma \to \varrho_\sigma$,

which clearly imply that $\mathsf{SPi} + \Sigma'_\Pi$ is complete.

(i) If $\Sigma'_\Pi \models_{\mathsf{Kr}} \varrho_\sigma \to \tau$ then $\Pi(\mathfrak{T}_\sigma) \models \varrho_\sigma \to \tau$. As $\Pi(\mathfrak{M}_\sigma), r_\sigma \models \varrho_\sigma$, we obtain that $\Pi(\mathfrak{M}_\sigma), r_\sigma \models \tau$, and so $\mathfrak{M}_{\varrho_\sigma}, r_\sigma \models \tau$ by Proposition 9. Now, take any Kripke model $\mathfrak{M}$ and a point w in it with $\mathfrak{M}, w \models \varrho_\sigma$. By Proposition 10, there is a homomorphism $h\colon \mathfrak{M}_{\varrho_\sigma} \to \mathfrak{M}$ with $h(r_\sigma) = w$, and so $\mathfrak{M}, w \models \tau$, as required.

(ii) As $\Pi(\mathfrak{M}_\sigma)$ extends $\mathfrak{M}_\sigma$, the identity map is a homomorphism from $\mathfrak{M}_\sigma$ to $\Pi(\mathfrak{M}_\sigma)$, from which $\vdash_{\mathsf{SLO}} \varrho_\sigma \to \sigma$ follows by Proposition 10. To prove that $\Sigma'_\Pi \vdash_{\mathsf{SLO}} \sigma \to \varrho_\sigma$, we construct $\Pi(\mathfrak{T}_\sigma)$ step-by-step as in (21)–(22). As every $\pi \in \Pi$ is forward-looking, the interim $\mathfrak{F}^i$ do not contain directed cycles, but they are not necessarily trees. However, as Π is stable, at each step the homomorphism $h^i\colon \mathfrak{G}^i \to \mathfrak{F}^i$ we use to obtain $\mathfrak{F}^{i+1}$ from $\mathfrak{F}^i$ is actually a $\mathfrak{G}^i \to \mathfrak{T}_\sigma$ homomorphism, and so we can arrange the steps in such a way that the depth of $h^i(u^i)$ in $\mathfrak{T}_\sigma$ is not smaller than the depth of $h^{i+1}(u^{i+1})$ in $\mathfrak{T}_\sigma$. This means that, for any $i < n$,

(29) there is a unique path in $\mathfrak{F}^i$ from r_σ to $h^i(u^i)$.

Let $\mathfrak{M}^i = (\mathfrak{F}^i, \mathfrak{v}_\sigma)$, for $i \le n$ (so $\mathfrak{M}^0 = \mathfrak{M}_\sigma$ and $\mathfrak{M}^n = \Pi(\mathfrak{M}_\sigma)$). We claim that

$$\Sigma \vdash_{\mathsf{SLO}} \mathsf{for}(\mathfrak{M}^i) \to \mathsf{for}(\mathfrak{M}^{i+1}), \quad \text{for every } i < n. \tag{30}$$

Indeed, fix some $i < n$ and suppose r^i is the root of $\mathfrak{G}^i$. By (29), $\mathsf{for}^{\mathfrak{M}^{i+1}}_{h^i(r^i)}$ differs from $\mathsf{for}^{\mathfrak{M}^i}_{h^i(r^i)}$ in an extra conjunct $\Diamond_{S^i}\mathsf{for}^{\mathfrak{M}^i}_{h^i(v^i)}$ at the unique place corresponding to the point $h^i(u^i)$. Therefore, the sp-implication $\mathsf{for}^{\mathfrak{M}^i}_{h^i(r^i)} \to \mathsf{for}^{\mathfrak{M}^{i+1}}_{h^i(r^i)}$ is in fact a substitution instance of $\boldsymbol{\iota}'_{\pi^i}$ obtained by replacing each p_x in $\boldsymbol{\iota}'_{\pi^i}$ with

$$\bigwedge_{h^i(x)\in\mathfrak{v}_\sigma(p)} p \;\wedge \bigwedge_{(y,R)\in A^i_x} \Diamond_R\, \mathsf{for}^{\mathfrak{M}^i}_y,$$

where

$$A^i_x = \big\{(y,R) \mid \big(h^i(x), y\big) \in R^{\mathfrak{F}^i}, \text{ but } y \ne h^i(x') \text{ for any } x' \text{ with } (x,x') \in R^{\mathfrak{G}^i}\big\}.$$

It remains to notice that $\big\{\mathsf{for}^{\mathfrak{M}^i}_{h^i(r^i)} \to \mathsf{for}^{\mathfrak{M}^{i+1}}_{h^i(r^i)}\big\} \vdash_{\mathsf{SLO}} \mathsf{for}(\mathfrak{M}^i) \to \mathsf{for}(\mathfrak{M}^{i+1})$, which proves (30). Finally, as

$$\mathsf{for}(\mathfrak{M}^0) = \mathsf{for}(\mathfrak{M}_\sigma) = \sigma \quad \text{and} \quad \mathsf{for}(\mathfrak{M}^n) = \mathsf{for}\big(\Pi(\mathfrak{M}_\sigma)\big) = \varrho_\sigma,$$

we obtain $\Sigma'_\Pi \vdash_{\mathsf{SLO}} \sigma \to \varrho_\sigma$, as required. ⊣

QUESTION 2. *Does Theorem 20 hold for* $\Sigma_\Pi = \{\boldsymbol{\iota}_\pi \mid \pi \in \Pi\}$ *in place of* Σ'_Π*?*

We do not know whether the spi-logics covered by Theorem 20 are complex. The next theorem indicates that showing this may require tricky embeddings.

THEOREM 21. *The spi-logic* $\mathsf{SPi} + \{\boldsymbol{\iota}'_{\pi_1}\}$ *with* π_1 *from Fig. 3 is complex.*

PROOF. Suppose $\mathfrak{A} = (A, \wedge, \top, \Diamond_R, \Diamond_S)$ is a SLO validating the sp-implication $\boldsymbol{\iota}'_{\pi_1} = \big(\Diamond_R(p \wedge \Diamond_R q) \to \Diamond_R(p \wedge \Diamond_S q)\big)$. Take the set $\mathcal{F}(\mathfrak{A})$ of all filters of $\mathfrak{A}$ and set, for $U, V \in \mathcal{F}(\mathfrak{A})$,

$$(U,V) \in R^{\mathfrak{G}} \iff \Diamond_R[V] \subseteq U, \text{ and } \Diamond_R a \in V \text{ implies } \Diamond_S a \in V \text{ for every } a;$$

$$(U,V) \in S^{\mathfrak{G}} \iff \Diamond_S[V] \subseteq U.$$

Then $\mathfrak{G} = (\mathcal{F}(\mathfrak{A}), R^{\mathfrak{G}}, S^{\mathfrak{G}})$ clearly validates Φ_{π_1}. Also, $S^{\mathfrak{G}}$ satisfies both (15) and (16), and $R^{\mathfrak{G}}$ satisfies (15). We show that $R^{\mathfrak{G}}$ satisfies (16) as well. Then, as shown in §4.1.2, $\mathfrak{A}$ would embed into $\mathfrak{G}^\star$. So suppose $\Diamond_R a \in U$ for some a. We need to find a $V \in \mathcal{F}(\mathfrak{A})$ such that $a \in V$ and $(U,V) \in R^{\mathfrak{G}}$. To this end, for any $X \subseteq A$, we let $X{\uparrow} = \{y \mid y \ge x \text{ for some } x \in X\}$, $V_0 = \{a\}{\uparrow}$ and, for every $n < \omega$,

$$V_{n+1} = \{x \wedge \Diamond_S y_1 \wedge \cdots \wedge \Diamond_S y_m \mid x \wedge \Diamond_R y_1 \wedge \cdots \wedge \Diamond_R y_m \in V_n\}{\uparrow}.$$

It can be shown by induction that, for every $n < \omega$,

- V_n is a filter;
- $\Diamond_R b \in V_n$ implies $\Diamond_S b \in V_{n+1}$, for every $b \in A$,
- $\Diamond_R[V_n] \subseteq U$.

We show that last item only. For $n = 0$, it holds because of the monotonicity of $\Diamond_R$. If $b \ge x \wedge \Diamond_S y_1 \wedge \cdots \wedge \Diamond_S y_m$, for some $x \wedge \Diamond_R y_1 \wedge \cdots \wedge \Diamond_R y_m \in V_n$, then by monotonicity and $\mathfrak{A} \models \Diamond_R(p \wedge \Diamond_R q) \le \Diamond_R(p \wedge \Diamond_S q)$, we have

$$\Diamond_R b \ge \Diamond_R(x \wedge \Diamond_S y_1 \wedge \cdots \wedge \Diamond_S y_m) \ge \Diamond_R(x \wedge \Diamond_R y_1 \wedge \cdots \wedge \Diamond_R y_m).$$

Since $\Diamond_R (x \wedge \Diamond_R y_1 \wedge \cdots \wedge \Diamond_R y_m) \in U$ by IH, $\Diamond_R b \in U$ follows.

As $V_0 \subseteq \cdots \subseteq V_n \subseteq \ldots$, their union $V = \bigcup_{n<\omega} V_n$ is the required filter.

Note that Theorem 21 cannot be proved using the simpler embedding of §4.1.1. Indeed, take the infinite SLO $\mathfrak{A} = (A, \wedge, \top, \Diamond)$ with the elements

$$\top = a_0 > a_1 > \cdots > a_n > \cdots > g,$$

$\Diamond_R g = \Diamond_S g = g$, $\Diamond_R a_n = \top$, and $\Diamond_S a_n = a_{n+1}$, for $n < \omega$. Then clearly $\mathfrak{A} \models \boldsymbol{\iota}'_{\pi_1}$. On the other hand, we claim that there are no $R^{\mathfrak{F}}, S^{\mathfrak{F}} \subseteq A \times A$ that both satisfy (12)–(13) and validate Φ_{π_1}. Indeed, suppose otherwise. As $a_0 \leq \Diamond_R a_0$, we have $(a_0, x) \in R^{\mathfrak{F}}$ for some $x \leq a_0$ by (13). As $a_0 \not\leq \Diamond_R g$, it follows by (12) that $x \neq g$, and so $(a_0, a_n) \in R^{\mathfrak{F}}$ for some $n < \omega$. As $a_n \leq \Diamond_R a_n$, we have $(a_n, y) \in R^{\mathfrak{F}}$ for some $y \leq a_n$ by (13). As $a_n \not\leq \Diamond_R g$, it follows by (12) that $y \neq g$, and so $(a_n, a_k) \in R^{\mathfrak{F}}$ for some k with $n \leq k < \omega$. Thus, Φ_{π_1} implies that $(a_n, a_k) \in S^{\mathfrak{F}}$, and so $a_n \leq \Diamond_S a_k$ by (12), which is a contradiction. ⊣

The next example shows that the stability condition is essential in Theorem 20.

Example 22. Consider the unstable set $\{\pi_3\}$ with the forward-looking profile π_3 from Fig. 3. It is easy to see that $\{\boldsymbol{\iota}'_{\pi_3}\} \models_{\mathsf{Kr}} \boldsymbol{\iota}$, where

$$\boldsymbol{\iota} = \big(\Diamond_S \big(q \wedge \Diamond_R (p \wedge \Diamond_R r)\big) \to \Diamond_S \big(q \wedge \Diamond_R (p \wedge \Diamond_S r)\big)\big).$$

On the other hand, the SLO in Fig. 4 validates $\boldsymbol{\iota}'_{\pi_3}$ but refutes $\boldsymbol{\iota}$ when q is $\{2\}$, p is $\{3,4\}$, and r is $\{5,6,7\}$. Therefore, $\mathsf{SPi} + \{\boldsymbol{\iota}'_{\pi_3}\}$ is not complete.

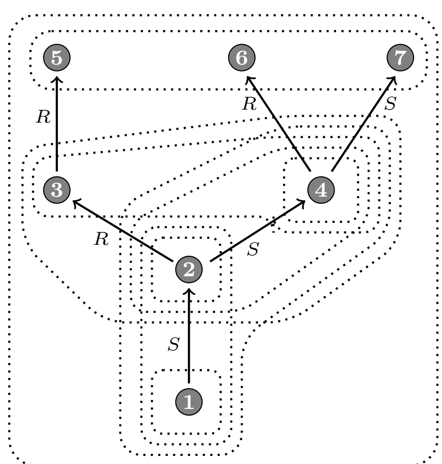


Figure 4. The SLO of Example 22.

However, Horn-implications with forward-looking but unstable profiles (such as $\boldsymbol{\iota}_{trans}$) can still axiomatise complex spi-logics. Likewise, spi-logics axiomatised by Horn-implications having non-forward-looking profiles such as $\boldsymbol{\iota}_{sym}$ can also be complete and even complex:

Theorem 23. *The following spi-logics are complex, and so complete:*

(*i*) $\mathsf{SPi} + \{\boldsymbol{\iota}_{sym}\}$;

(*ii*) $\mathsf{SPi}_{equiv} = \mathsf{SPi} + \Sigma_{equiv} = \mathsf{SPi} + \Sigma'_{equiv}$, *where* $\Sigma_{equiv} = \{\boldsymbol{\iota}_{refl}, \boldsymbol{\iota}_{trans}, \boldsymbol{\iota}_{sym}\}$ *and* $\Sigma'_{equiv} = \{\boldsymbol{\iota}_{refl}, \boldsymbol{\iota}_{trans}, \boldsymbol{\iota}_{eucl}\}$.

Proof. (*i*) Let $\mathfrak{A} = (A, \wedge, \top, \Diamond)$ be a SLO with $\mathfrak{A} \models \boldsymbol{\iota}_{sym}$. For $a, b \in A$, let

$$(a, b) \in R^{\mathfrak{F}} \iff a \leq \Diamond b \text{ and } b \leq \Diamond a. \tag{31}$$

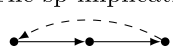
Then $R^{\mathfrak{F}}$ is clearly symmetric and satisfies (12). We show that it satisfies (13) as well, and so, as shown in §4.1.1, $\mathfrak{A}$ embeds to $\mathfrak{F}^\star$, for $\mathfrak{F} = (A, R^{\mathfrak{F}})$. To this end, fix some $a \in A$ and let x be such that $a \leq \Diamond x$. Then, by $\mathfrak{A} \models \boldsymbol{\iota}_{sym}$, we have

$$a = a \wedge \Diamond x \leq \Diamond\big(\Diamond a \wedge x\big).$$

Let $b = \Diamond a \wedge x$. Then $a \leq \Diamond b$, $b \leq x$ and $b \leq \Diamond a$; so $(a, b) \in R^{\mathfrak{F}}$, as required in (13).

(*ii*) It is easy to see that $\{\boldsymbol{\iota}_{refl}, \boldsymbol{\iota}_{eucl}\} \vdash_{\mathsf{SLO}} \boldsymbol{\iota}_{sym}$ and $\{\boldsymbol{\iota}_{trans}, \boldsymbol{\iota}_{sym}\} \vdash_{\mathsf{SLO}} \boldsymbol{\iota}_{eucl}$, and so $\mathsf{SPi} + \Sigma_{equiv} = \mathsf{SPi} + \Sigma'_{equiv}$. It is straightforward to check that if $\mathfrak{A} \models \boldsymbol{\iota}_{refl}$ and $\mathfrak{A} \models \boldsymbol{\iota}_{trans}$, then the $R^{\mathfrak{F}}$ defined in (31) is reflexive and transitive as well. Note that Jackson [46] proves completeness of SPi_{equiv} by showing that $\Sigma'_{equiv} \models_{\mathsf{BAO}}$ is conservative over $\Sigma'_{equiv} \models_{\mathsf{SLO}}$. ⊣

The next two examples show incomplete spi-logics axiomatised by sp-implications with non-rooted, non-forward looking and unstable tree-profiles.

Example 24. The sp-implication $\boldsymbol{\iota} = \big(q \wedge \Diamond\Diamond p \to \Diamond\Diamond(p \wedge \Diamond q)\big)$ has the non-rooted tree-profile  . It is easy to see that $\{\boldsymbol{\iota}\} \models_{\mathsf{Kr}} \Diamond\Diamond\Diamond\Diamond p \to \Diamond p$. On the other hand, the SLO in Fig. 5 validates $\boldsymbol{\iota}$ but refutes $\Diamond\Diamond\Diamond\Diamond p \to \Diamond p$ when p is $\{5\}$. Therefore, $\mathsf{SPi} + \{\boldsymbol{\iota}\}$ is not complete.

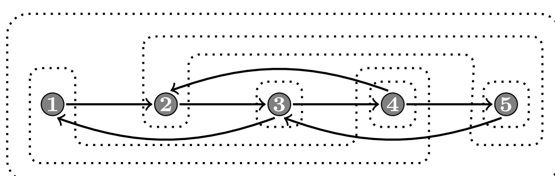
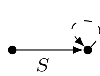
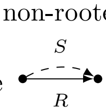


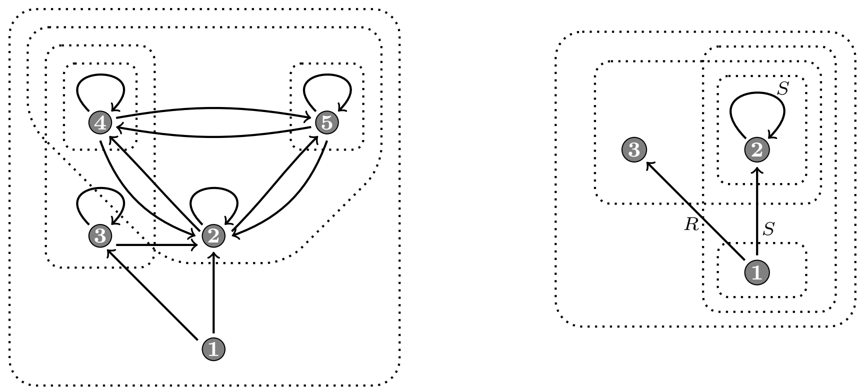
Figure 5. The SLO of Example 24.

Example 25. Consider next the sp-implication $\boldsymbol{\iota}_{eucl}$ (see Table 3). It is not hard to see that $\{\boldsymbol{\iota}_{eucl}\} \models_{\mathsf{Kr}} \Diamond\Diamond p \wedge \Diamond q \to \Diamond(q \wedge \Diamond p)$. On the other hand, the SLO in Fig. 6 (a) validates $\boldsymbol{\iota}_{eucl}$ but refutes $\Diamond\Diamond p \wedge \Diamond q \to \Diamond(q \wedge \Diamond p)$ when p is $\{5\}$ and q is $\{3, 4\}$. Therefore, $\mathsf{SPi} + \{\boldsymbol{\iota}_{eucl}\}$ is not complete.

Example 26 ([9]). Consider $\boldsymbol{\iota} = \big(\Diamond_S\, p \to \Diamond_S\, (p \wedge \Diamond_S\, p)\big)$ with non-rooted tree-profile 

and $\boldsymbol{\iota}' = (\Diamond_R\, p \to \Diamond_S\, p)$ with rooted tree-profile 

. Then $\{\boldsymbol{\iota}, \boldsymbol{\iota}'\} \models_{\mathsf{Kr}} \Diamond_R\, p \to \Diamond_R\, (p \wedge \Diamond_S\, p)$. However, the SLO in Fig. 6 (b) validates both $\boldsymbol{\iota}$ and $\boldsymbol{\iota}'$, but refutes $\Diamond_R\, p \to \Diamond_R\, (p \wedge \Diamond_S\, p)$ when p is $\{2, 3\}$. Therefore, $\mathsf{SPi} + \{\boldsymbol{\iota}, \boldsymbol{\iota}'\}$ is not complete.



(a) (b)

FIGURE 6. The SLOs of Examples 25 and 26.

This example generalises to the following theorem:

THEOREM 27. *For any Horn-implication $\boldsymbol{\iota}$ with a non-rooted tree-profile, there is a Horn-implication $\boldsymbol{\iota}'$ with a rooted tree-profile (and a fresh diamond operator) such that the spi-logic* $\mathsf{SPi} + \{\boldsymbol{\iota}, \boldsymbol{\iota}'\}$ *is not complete.*

PROOF. Suppose $\pi = (\mathfrak{G}, S, u, v)$ is the non-rooted profile of $\boldsymbol{\iota} = (\sigma \to \tau)$. Denote by r the root of $\mathfrak{G} = (\Delta, R^{\mathfrak{G}})_{R \in \mathcal{R}}$ and by w the successor of r on the branch from r to u with, say, $(r, w) \in R^{\mathfrak{G}}$ for some $R \in \mathcal{R}$. Define $\mathfrak{G}'$ to be a tree whose points are copies x' of the points x in $\mathfrak{G}$, and the arrows between them are the same as in $\mathfrak{G}$ except that we replace the $R^{\mathfrak{G}'}$-arrow from r' to w' with an $R_\dagger^{\mathfrak{G}'}$-arrow, for some fresh $R_\dagger \notin \mathcal{R}$. Let $\pi' = (\mathfrak{G}', S, u', v')$ and let $\boldsymbol{\iota}' = (\Diamond_{R_\dagger} p \to \Diamond_R p)$. It is readily seen that any frame validating $\{\boldsymbol{\iota}, \boldsymbol{\iota}'\}$ also validates the sp-implication $\boldsymbol{\iota}_{\pi'}$.

We now construct a SLO $\mathfrak{A}$ validating $\{\boldsymbol{\iota}, \boldsymbol{\iota}'\}$ but refuting $\boldsymbol{\iota}_{\pi'}$. Consider the Horn closure $\pi(\mathfrak{G})$ of $\mathfrak{G}$. Clearly, $\pi(\mathfrak{G}) \models \Phi_\pi$, from which $\pi(\mathfrak{G}) \models \Psi_{\boldsymbol{\iota}}$ and

$$\pi(\mathfrak{G}) \models \boldsymbol{\iota}. \tag{32}$$

Now let $\mathfrak{F}$ be the result of merging the roots r of $\pi(\mathfrak{G})$ and r' of $\mathfrak{G}'$ into a single point. We define $\mathfrak{A}$ as the subalgebra of $\mathfrak{F}^\star$ with domain

$$A = \{X \cup X' \mid X \subseteq \mathfrak{G},\ X' \subseteq \mathfrak{G}',\ X' \subseteq' X\},$$

where $X' \subseteq' X$ iff $x' \in X'$ implies $x \in X$. Then $\emptyset$ and the domain of $\mathfrak{F}$ clearly belong to A. Also, A is closed under intersections because we clearly have $(X \cup X') \cap (Y \cup Y') = (X \cap Y) \cup (X' \cap Y')$; here we use the fact that $r = r'$. Furthermore, $\Diamond^+_{R_\dagger}(X \cup X') = \emptyset$ if $w' \notin X'$, $\Diamond^+_{R_\dagger}(X \cup X') = \{r\}$ if $w' \in X'$, and $\Diamond^+_Q(X \cup X') = \Diamond^+_Q X \cup \Diamond^+_Q X'$ with $\Diamond^+_Q X' \subseteq' \Diamond^+_Q X$, for any Q different from $R_\dagger$. Thus, $\mathfrak{A}$ is a SLO. Observe also that, for every $X \cup X' \in A$, we have $\Diamond^+_{R_\dagger}(X \cup X') \subseteq \Diamond^+_R(X \cup X')$, and so $\mathfrak{A} \models \boldsymbol{\iota}'$.

Next, we show that $\mathfrak{A} \not\models \boldsymbol{\iota}_{\pi'}$. Indeed, suppose $\boldsymbol{\iota}_{\pi'} = (\alpha \to \Diamond_{R_\dagger} \beta)$ (cf. (25)). We have $\mathfrak{G}' \not\models \boldsymbol{\iota}_{\pi'}$ by (20), and so there exist a Kripke model $\mathfrak{M} = (\mathfrak{G}', \mathfrak{v})$ and some w in it such that $\mathfrak{M}, w \models \alpha$ but $\mathfrak{M}, w \not\models \Diamond_{R_\dagger} \beta$. We define a valuation $\mathfrak{a}$ in $\mathfrak{A}$ by taking

$$\mathfrak{a}(p) = \mathfrak{v}(p) \cup \{x \mid x' \in \mathfrak{v}(p)\}, \quad \text{for every variable } p.$$

It is easy to see that $\varrho[\mathfrak{a}] \cap \Delta = \{w \mid \mathfrak{M}, w \models \varrho\}$, for every sp-formula ϱ. Then $\alpha[\mathfrak{a}] \supseteq \{w \mid \mathfrak{M}, w \models \alpha\}$ and $(\Diamond_{R_\dagger} \beta)[\mathfrak{a}] = \Diamond^+_{R_\dagger}\big(\beta[a]\big) = \Diamond^+_{R_\dagger}\big(\{w \mid \mathfrak{M}, w \models \beta\}\big) = \{w \mid \mathfrak{M}, w \models \Diamond_{R_\dagger} \beta\}$, from which $\mathfrak{A} \not\models \boldsymbol{\iota}_{\pi'}[\mathfrak{a}]$.

It remains to establish $\mathfrak{A} \models \boldsymbol{\iota}$. As $\mathfrak{A}$ is a subalgebra of $\mathfrak{F}^\star$, it is enough to show that $\mathfrak{F} \models \boldsymbol{\iota}$. Take any Kripke model $\mathfrak{M} = (\mathfrak{F}, \mathfrak{v})$ and suppose $\mathfrak{M}, x \models \sigma$, for some point x in $\mathfrak{F}$. By Proposition 7, there is a homomorphism $h\colon \mathfrak{M}_\sigma \to \mathfrak{M}$ with $h(r_\sigma) = x$ for the root r_σ of $\mathfrak{T}_\sigma$. We show that $\mathfrak{M}, x \models \tau$. Indeed, note first that x cannot be a non-root point in $\mathfrak{G}'$ because otherwise we would have a homomorphism $f\colon \mathfrak{T}_\sigma \to \mathfrak{G}$ with $f(r_\sigma) \neq r$, contradicting Proposition 14 (*ii*). Thus, x is a point in $\pi(\mathfrak{G})$. We define a map $h'\colon \mathfrak{T}_\sigma \to \pi(\mathfrak{G})$ by taking

$$h'(y) = \begin{cases} h(y), & \text{if } h(y) \text{ is in } \pi(\mathfrak{G}),\\ z, & \text{if } h(y) = z' \text{ for some } z' \text{ in } \mathfrak{G}'.\end{cases}$$

As σ does not contain $\Diamond_{R_\dagger}$, it is easy to see that h' is a homomorphism from $\mathfrak{M}_\sigma$ to the Kripke model $\mathfrak{M}^- = \big(\pi(\mathfrak{G}), \mathfrak{v} \upharpoonright \pi(\mathfrak{G})\big)$ with $h'(r_\sigma) = h(r_\sigma) = x$, and so $\mathfrak{M}^-, x \models \sigma$ by Proposition 7. Then we have $\mathfrak{M}^-, x \models \tau$ and so $\mathfrak{M}, x \models \tau$ by (32), as required. ⊣

5.2. Universal Horn correspondents with equality. Example 1 showed that the spi-logic $\mathsf{SPi} + \{\Diamond p \to p\}$ with the correspondent

$$\forall x, y\, (R(x, y) \to (x = y))$$

is incomplete. It is easy to find an extension of this spi-logic that is complex:

THEOREM 28. *The spi-logic* $\mathsf{SPi} + \{\boldsymbol{\iota}_{\mathit{refl}}, \Diamond p \to p\} = \mathsf{SPi} + (\Sigma_{qo} \cup \{\Diamond p \to p\}) = \mathsf{SPi} + (\Sigma_{\mathit{equiv}} \cup \{\Diamond p \to p\}) = \mathsf{SPi} + (\Sigma'_{\mathit{equiv}} \cup \{\Diamond p \to p\})$ *is complex, and so complete.*

PROOF. It is easy to see that $\{\Diamond p \to p\} \vdash_{\mathsf{SLO}} \boldsymbol{\iota}_{\mathit{trans}}$ and $\{\boldsymbol{\iota}_{\mathit{refl}}, \Diamond p \to p\} \vdash_{\mathsf{SLO}} \boldsymbol{\iota}_{\mathit{eucl}}$, and so all four spi-logics are the same. The correspondent of this spi-logic is

$$\Phi = \forall x, y\, \big(R(x, y) \leftrightarrow x = y\big). \tag{33}$$

Let $\mathfrak{A} = (A, \wedge, \top, \Diamond)$ be a SLO with $\mathfrak{A} \models \{\boldsymbol{\iota}_{\mathit{refl}}, \Diamond p \to p\}$. For all $a, b \in A$, we set $(a, b) \in R^{\mathfrak{F}}$ iff $a = b$. Then $R^{\mathfrak{F}}$ clearly satisfies Φ, (12) and (13). So, as is shown in §4.1.1, $\mathfrak{A}$ embeds to $\mathfrak{F}^\star$ for $\mathfrak{F} = (A, R^{\mathfrak{F}})$. ⊣

Our next example is the sp-implication

$$\boldsymbol{\iota}_{\mathit{fun}} = \big(\Diamond p \wedge \Diamond q \to \Diamond(p \wedge q)\big)$$

saying that $\Diamond$ is a semilattice homomorphism.[7] The first-order correspondent of $\boldsymbol{\iota}_{\mathit{fun}}$ is *functionality*:

$$\forall x, y, z\, \big(R(x, y) \wedge R(x, z) \to y = z\big).$$

[7] In [46], any $\mathfrak{A} \in \mathsf{SLO}_{\mathsf{SPi}_{qo}}$ validating $\boldsymbol{\iota}_{\mathit{fun}}$ is called *entropic*.

It is easy to see that $\{\boldsymbol{\iota}_{\mathit{refl}}, \Diamond p \to p\} \vdash_{\mathsf{SLO}} \boldsymbol{\iota}_{\mathit{fun}}$ and $\{\boldsymbol{\iota}_{\mathit{refl}}, \boldsymbol{\iota}_{\mathit{fun}}\} \vdash_{\mathsf{SLO}} \boldsymbol{\iota}_{\mathit{eucl}}$, and so

$$\mathsf{SPi} + \{\boldsymbol{\iota}_{\mathit{refl}}, \boldsymbol{\iota}_{\mathit{fun}}, \Diamond p \to p\} = \mathsf{SPi} + (\Sigma_{qo} \cup \{\boldsymbol{\iota}_{\mathit{fun}}, \Diamond p \to p\}) = \mathsf{SPi} + (\Sigma_{\mathit{equiv}} \cup \{\boldsymbol{\iota}_{\mathit{fun}}, \Diamond p \to p\}) = \mathsf{SPi} + (\Sigma'_{\mathit{equiv}} \cup \{\boldsymbol{\iota}_{\mathit{fun}}, \Diamond p \to p\})$$

is the same spi-logic as in Theorem 28.

THEOREM 29. (i) *The spi-logic* $\mathsf{SPi} + \{\boldsymbol{\iota}_{\mathit{fun}}\}$ *is complex, and so complete.*
On the other hand, the following spi-logics are incomplete:

(ii) $\mathsf{SPi} + \{\Diamond p \to p, \boldsymbol{\iota}_{\mathit{fun}}\}$;
(iii) $\mathsf{SPi} + \{\boldsymbol{\iota}_{\mathit{refl}}, \boldsymbol{\iota}_{\mathit{fun}}\} = \mathsf{SPi} + \{\boldsymbol{\iota}_{\mathit{refl}}, \boldsymbol{\iota}_{\mathit{eucl}}, \boldsymbol{\iota}_{\mathit{fun}}\}$;
(iv) $\mathsf{SPi} + (\Sigma_{qo} \cup \{\boldsymbol{\iota}_{\mathit{fun}}\}) = \mathsf{SPi} + (\Sigma_{\mathit{equiv}} \cup \{\boldsymbol{\iota}_{\mathit{fun}}\}) = \mathsf{SPi} + (\Sigma'_{\mathit{equiv}} \cup \{\boldsymbol{\iota}_{\mathit{fun}}\})$;
(v) $\mathsf{SPi} + \{\boldsymbol{\iota}_{\mathit{sym}}, \boldsymbol{\iota}_{\mathit{fun}}\}$.

PROOF. (i) Let $\mathfrak{A} = (A, \wedge, \top, \Diamond)$ be a SLO such that $\mathfrak{A} \models \boldsymbol{\iota}_{\mathit{fun}}$ and let $\mathcal{F}(\mathfrak{A})$ be the set of all filters of $\mathfrak{A}$. We claim that in this case $\Diamond^{-1}[U]$ is either empty or a filter, for every $U \in \mathcal{F}(\mathfrak{A})$. Indeed, $\Diamond^{-1}[U]$ is up-closed by the monotonicity of $\Diamond$, and $\wedge$-closed by $\boldsymbol{\iota}_{\mathit{fun}}$. Now we set, for $U, V \in \mathcal{F}(\mathfrak{A})$,

$$(U, V) \in R^{\mathfrak{G}} \quad \Longleftrightarrow \quad V = \Diamond^{-1}[U].$$

Then $R^{\mathfrak{G}}$ is clearly functional and satisfies (15). It is readily seen that it satisfies (16) as well. So, as shown in §4.1.2, $\mathfrak{A}$ embeds into $\mathfrak{G}^\star$ for $\mathfrak{G} = (\mathcal{F}(\mathfrak{A}), R^{\mathfrak{G}})$.

(ii) The proof in Example 1 again works. Note that the SLO in Fig. 7 shows that the spi-logics $\mathsf{SPi} + \{\Diamond p \to p, \boldsymbol{\iota}_{\mathit{fun}}\}$ and $\mathsf{SPi} + \{\Diamond p \to p\}$ are not the same.

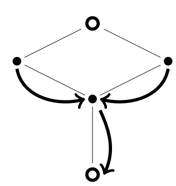


FIGURE 7. A SLO showing that $\{\Diamond p \to p\} \not\vdash_{\mathsf{SLO}} \boldsymbol{\iota}_{\mathit{fun}}$.

(iii) The correspondent of this spi-logic is Φ in (33). So it is easy to see that $\{\boldsymbol{\iota}_{\mathit{refl}}, \boldsymbol{\iota}_{\mathit{fun}}\} \models_{\mathsf{Kr}} \boldsymbol{\iota}_{\mathit{trans}}$. On the other hand, take the SLO $\mathfrak{A}$ with 3 elements $b \le a \le \top$, $\Diamond b = a$ and $\Diamond a = \Diamond \top = \top$. Then $\mathfrak{A} \models \boldsymbol{\iota}_{\mathit{refl}}$ and $\mathfrak{A} \models \boldsymbol{\iota}_{\mathit{fun}}$, but $\Diamond\Diamond b \not\le \Diamond b$.

(iv) The correspondent of this spi-logic is again Φ in (33). So it is easy to see that $\Sigma_{qo} \cup \{\boldsymbol{\iota}_{\mathit{fun}}\} \models_{\mathsf{Kr}} \Diamond p \to p$. On the other hand, take the SLO $\mathfrak{A}$ with 3 elements $b \le a \le \top$, $\Diamond b = b$ and $\Diamond a = \Diamond \top = \top$. Then $\mathfrak{A} \models \Sigma_{qo} \cup \{\boldsymbol{\iota}_{\mathit{fun}}\}$, but $\Diamond a = \top \not\le a$.

(v) It is easy to see that $\Diamond\Diamond p \to p$ is valid in any symmetric and functional frame, and so $\{\boldsymbol{\iota}_{\mathit{sym}}, \boldsymbol{\iota}_{\mathit{fun}}\} \models_{\mathsf{Kr}} \Diamond\Diamond p \to p$. On the other hand, in the SLO $\mathfrak{A}$ from item (iv), $\Diamond\Diamond a = \top \not\le a$. ⊣

REMARK 30. Theorem 29 (i) cannot be proved using the simpler embedding of §4.1.1. Indeed, take the SLO $\mathfrak{A} = (A, \wedge, \top, \Diamond)$ where $A = \{a_n \mid n < \omega\}$, $a_n \wedge a_m = a_n$ whenever $n \ge m$ (and so $\top = a_0$), and $\Diamond a_n = \top$ for all $n < \omega$. Then $\mathfrak{A} \models \boldsymbol{\iota}_{\mathit{fun}}$ clearly holds. On the other hand, we claim that there is no functional $R^{\mathfrak{F}} \subseteq A \times A$ satisfying (13). Indeed, suppose $R^{\mathfrak{F}}$ satisfies (13). Since for every $n < \omega$, we have $\top \leq \Diamond a_n$, it follows from (13) that, for any $n < \omega$, there exists $m \geq n$ such that $(\top, a_m) \in R^{\mathfrak{F}}$, and so $R^{\mathfrak{F}}$ is not functional.

5.3. Negative universal Horn correspondents. Finally, we discuss sp-implications with Horn correspondents of the form 'false' and 'something implies false'. Recall that Example 2 showed that the spi-logic $\mathsf{SPi}+\{\Diamond p \to \Diamond q\}$ with the correspondent $R = \emptyset$—or $\forall x, y\,(R(x,y) \to \bot)$, to be more precise—is incomplete. The next theorem gives an incomplete extension of this spi-logic:

THEOREM 31. *The spi-logic* $\mathsf{SPi} + \{\boldsymbol{\iota}_{refl}, \Diamond p \to \Diamond q\} = \mathsf{SPi} + (\Sigma_{qo} \cup \{\Diamond p \to \Diamond q\}) = \mathsf{SPi} + (\Sigma_{equiv} \cup \{\Diamond p \to \Diamond q\})$ *is incomplete.*

PROOF. It is easy to see that $\{\Diamond p \to \Diamond q\} \vdash_{\mathsf{SLO}} \boldsymbol{\iota}_{trans}$ and $\{\Diamond p \to \Diamond q\} \vdash_{\mathsf{SLO}} \boldsymbol{\iota}_{sym}$, and so all three spi-logics are the same. As there is no frame validating $\Diamond p \to \Diamond q$, we have $\{\boldsymbol{\iota}_{refl}, \Diamond p \to \Diamond q\} \models_{\mathsf{Kr}} \Diamond\top \to p$. On the other hand, we have $\{\boldsymbol{\iota}_{refl}, \Diamond p \to \Diamond q\} \not\models_{\mathsf{SLO}} \Diamond\top \to p$, as the SLO $\mathfrak{A}$ with 2 elements $a \leq \top$ such that $\Diamond a = \Diamond\top = \top$ validates both $\boldsymbol{\iota}_{refl}$ and $\Diamond p \to \Diamond q$, but refutes $\Diamond\top \to p$, since $\Diamond\top = \top \not\leq a$. ⊣

Of course, not every spi-logic without frames is incomplete. We call an spi-logic L *trivial* if $(p \to q) \in L$. Then we clearly have:

PROPOSITION 32. *Every trivial spi-logic is complete.*

The following two theorems imply that spi-logics axiomatised by sp-implications of the form $\Diamond_R \Diamond_S\, p \to q$ (with negative universal Horn correspondent $\forall x, y, z\,\big((R(x,y) \wedge S(y,z) \to \bot)\big)$ behave differently in the uni- and multi-modal cases.

THEOREM 33. $\mathsf{SPi} + \{\Diamond^n p \to q\}$ *is complex, and so complete, for any* $n \geq 1$.

PROOF. The correspondent of $\{\Diamond^n p \to q\}$ is 'there is no R-chain of length n'. Let $\mathfrak{A} = (A, \wedge, \top, \Diamond)$ be a SLO with $\mathfrak{A} \models \Diamond^n p \to q$. Then $\Diamond^n\top$ is the $\leq$-smallest element in $\mathfrak{A}$. If $|A| = 1$ then $\mathfrak{A}$ is clearly embeddable into $\mathfrak{F}^\star$ of any frame $\mathfrak{F}$. So let $|A| > 1$ and $A^- = A \setminus \{\Diamond^n\top\}$. For any $a, b \in A^-$, let $(a,b) \in R^{\mathfrak{F}}$ iff $a \leq \Diamond b$, and let $\mathfrak{F} = (A^-, R^{\mathfrak{F}})$. Then $R^{\mathfrak{F}}$ clearly satisfies (12). As $\{\Diamond^n p \to q\} \vdash_{\mathsf{SLO}} \Diamond^{n+1}\top \to \Diamond^n\top$, we also have the following analogue of (13):

$$\forall a \in A^-, b \in A\ \big[a \leq \Diamond_R\, b\ \Rightarrow\ \exists c \in A^-\ \big(c \leq b\ \text{ and }\ (a,c) \in R^{\mathfrak{F}}\big)\big].$$

A proof similar to the one in §4.1.1 shows that the map $\eta(a) = \{b \in A^- \mid b \leq a\}$ embeds $\mathfrak{A}$ into $\mathfrak{F}^\star$. Now, suppose $\mathfrak{F}$ contains an $R^{\mathfrak{F}}$-chain of length n, that is, there are $a_0, a_1, \dots, a_n \in A^-$ with $(a_i, a_{i+1}) \in R^{\mathfrak{F}}$ for all $i < n$. Then $a_0 \leq \Diamond^n a_n$, and so $a_0 = \Diamond^n\top$, contrary to $a_0 \in A^-$. ⊣

THEOREM 34. *Let σ be an sp-formula containing $\Diamond_S$ but not $\Diamond_R$, and let q be a propositional variable not occurring in σ. Then* $\mathsf{SPi}+\{\Diamond_R\, \sigma \to q\}$ *is incomplete.*

PROOF. It is easy to see that $\{\Diamond_R\, \sigma \to q\} \models_{\mathsf{Kr}} \Diamond_S \Diamond_R\, \sigma \to \Diamond_R\, \sigma$ for any $\Diamond_S$ in σ. On the other hand, take the SLO $\mathfrak{A}$ with 2 elements $a \leq \top$ such that $\Diamond_R\, a = \Diamond_R \top = a$ and $\Diamond_S\, a = \Diamond_S \top = \top$, for $S \neq R$. Then $\mathfrak{A}$ validates $\Diamond_R\, \sigma \to q$ but refutes $\Diamond_S \Diamond_R\, \sigma \to \Diamond_R\, \sigma$, and so $\mathsf{SPi} + \{\Diamond_R\, \sigma \to q\}$ is incomplete. ⊣

§6. Completeness of spi-logics with existential correspondents. We now extend Theorem 15 to sp-implications whose correspondents contain existential quantifiers (but no disjunction) on the right-hand side of implication.

It is not hard to see, using distributivity of $\wedge$ over $\vee$, that the correspondent $\Psi_{\boldsymbol{\iota}}$ of an sp-implication $\boldsymbol{\iota} = (\sigma \to \tau)$ (see (18) and (19)) can be equivalently rewritten as

$$(34)\quad \Psi_{\boldsymbol{\iota}} \;=\; \forall \hat{v}_0, \ldots, \hat{v}_{n_\sigma} \Big(\bigwedge_{\substack{i,j \le n_\sigma,\, R \in \mathcal{R} \\ (v_i, v_j) \in R_\sigma}} R(\hat{v}_i, \hat{v}_j) \to$$
$$\exists \hat{u}_0, \ldots, \hat{u}_{n_\tau} \big((\hat{v}_0 = \hat{u}_0) \wedge \bigwedge_{\substack{i,j \le n_\tau,\, R \in \mathcal{R} \\ (u_i, u_j) \in R_\tau}} R(\hat{u}_i, \hat{u}_j) \wedge \bigvee_{f \in Y_{\sigma,\tau}} \bigwedge_{\substack{(u_i, p) \in X_\tau \\ f(u_i, p) = v_j}} (\hat{u}_i = \hat{v}_j) \big) \Big),$$

where $X_\tau = \{(u_i, p) \mid p \text{ is a variable and } u_i \in \mathfrak{v}_\tau(p)\}$ and

$$Y_{\sigma,\tau} = \{f \mid f \colon X_\tau \to W_\sigma, f(u_i, p) \in \mathfrak{v}_\sigma(p) \text{ for all } (u_i, p) \in X_\tau\}.$$

If the right-hand side of $\Psi_{\boldsymbol{\iota}}$ does not contain any disjunction, this means that $Y_{\sigma,\tau}$ consists of a single 'choice' function f.

THEOREM 35. *Any spi-logic axiomatised by sp-implications $\sigma \to \tau$ such that*

(*i*) *every variable in τ occurs in σ exactly once,*
(*ii*) *$|W_\tau| \ge 2$ and all points in any $\mathfrak{v}_\tau(p)$ are leaves of $\mathfrak{T}_\tau$,*
(*iii*) *$\mathfrak{v}_\tau(p) \cap \mathfrak{v}_\tau(q) = \emptyset$ whenever $p \neq q$*

is complex, and so complete.

PROOF. Suppose that $\boldsymbol{\iota} = (\sigma \to \tau)$ and the points of $W_\sigma = \{v_0, \ldots, v_{n_\sigma}\}$ and $W_\tau = \{u_0, \ldots, u_{n_\tau}\}$ are listed so that $(v_i, v_j) \in R_\sigma$ or $(u_i, u_j) \in R_\tau$ imply $i < j$ (and so $v_0 = r_\sigma$ and $u_0 = r_\tau$). By (34) and (*i*),

$$\Psi_{\boldsymbol{\iota}} \;=\; \forall \hat{v}_0, \ldots, \hat{v}_{n_\sigma} \Big(\bigwedge_{\substack{i,j \le n_\sigma,\, R \in \mathcal{R} \\ (v_i, v_j) \in R_\sigma}} R(\hat{v}_i, \hat{v}_j) \to$$
$$\exists \hat{u}_0, \ldots, \hat{u}_{n_\tau} \big((\hat{v}_0 = \hat{u}_0) \wedge \bigwedge_{\substack{i,j \le n_\tau,\, R \in \mathcal{R} \\ (u_i, u_j) \in R_\tau}} R(\hat{u}_i, \hat{u}_j) \wedge \bigwedge_{\substack{u_i \in \mathfrak{v}_\tau(p) \\ \mathfrak{v}_\sigma(p) = \{v_j\}}} (\hat{u}_i = \hat{v}_j) \big) \Big).$$

Let $\mathfrak{A} = (A, \wedge, \top, \Diamond_R)_{R \in \mathcal{R}}$ be a SLO validating $\boldsymbol{\iota}$. It is shown in §4.1.1 that $\mathfrak{A}$ can be embedded into $\mathfrak{F}^\star$ for the frame $\mathfrak{F} = (A, R^{\mathfrak{F}})_{R \in \mathcal{R}}$ with $R^{\mathfrak{F}}$ defined by (14). We claim that $\mathfrak{F} \models \Psi_{\boldsymbol{\iota}}$. Indeed, for each point v_i in $\mathfrak{T}_\sigma$, take some $a_i \in A$ such that $(v_i, v_j) \in R_\sigma$ imply $(a_i, a_j) \in R^{\mathfrak{F}}$, that is, $a_i \le \Diamond_R\, a_j$. We need to find $b_0, \ldots, b_m \in A$ such that $b_0 = a_0$ and the following properties hold, for $j = 0, \ldots, m$:

(a) $b_j = a_k$ if $u_j \in \mathfrak{v}_\tau(p)$ and $\mathfrak{v}_\sigma(p) = \{v_k\}$, for some variable p;
(b) if $(u_j, u_k) \in R_\tau$ then $(b_j, b_k) \in R^{\mathfrak{F}}$, that is, $b_j \le \Diamond_R\, b_k$.

We define inductively $b_m, \dots, b_0$ by taking:

$$b_j = \begin{cases} a_k, & \text{if } u_j \in \mathfrak{v}_\tau(p),\ \mathfrak{v}_\sigma(p) = \{v_k\} \text{ for some } p,\\ \top, & \text{if } u_j \text{ is a leaf and there is no } p \text{ with } u_j \in \mathfrak{v}_\tau(p),\\ \Diamond_{R^1} b_{k_1} \wedge \dots \wedge \Diamond_{R^\ell} b_{k_\ell}, & \text{if } j \neq 0 \text{ and } u_j \text{ has } \ell > 0 \text{ successors}\\ & \qquad u_{k_1}, \dots, u_{k_\ell} \text{ with } (u_j, u_{k_i}) \in R^i_\tau,\\ a_0, & \text{if } j = 0. \end{cases}$$

By (i)–(iii), b_j is well-defined. We clearly have (a) and (b), for $j \neq 0$. To show (b) for $j = 0$, take the following valuation $\mathfrak{a}$ in $\mathfrak{A}$, for all variables p:

$$\mathfrak{a}(p) = \begin{cases} a_k, & \text{if } p \text{ occurs in } \sigma \text{ and } \mathfrak{v}_\sigma(p) = \{v_k\},\\ \top, & \text{otherwise.} \end{cases}$$

By (i), $\mathfrak{a}$ is well-defined. Let $\tau_j = \mathsf{for}^{\mathfrak{M}_\tau}_{u_j}$, for $j = m, \dots, 1$ (cf. §4.2.2). We prove that

$$\tau_j[\mathfrak{a}] \le b_j, \qquad \text{for every } j = m, \dots, 1. \tag{35}$$

Indeed, if u_j is a leaf, then either $\tau_j = \top = b_j$, or $\tau_j = p$ for some variable p, and so $\tau_j[\mathfrak{a}] = a_k = b_j$ for k with $\mathfrak{v}_\sigma(p) = \{v_k\}$. Now suppose inductively that, for some $j \ge 1$, (35) holds for every k with $j < k \le m$. If u_j has $\ell > 0$ successors $u_{k_1}, \dots, u_{k_\ell}$ with $(u_j, u_{k_i}) \in R^i_\tau$, then each $\Diamond_{R^i} \tau_{k_i}$ is a conjunct of τ_j, and so, by IH and monotonicity,

$$\tau_j[\mathfrak{a}] \le \Diamond_{R^1} \tau_{k_1}[\mathfrak{a}] \wedge \dots \wedge \Diamond_{R^\ell} \tau_{k_\ell}[\mathfrak{a}] \le \Diamond_{R^1} b_{k_1} \wedge \dots \wedge \Diamond_{R^\ell} b_{k_\ell} = b_j,$$

as required in (35). Next, let $\sigma_i = \mathsf{for}^{\mathfrak{M}_\sigma}_{v_i}$, $i = 0, \dots, n$. We prove that

$$a_i \le \sigma_i[\mathfrak{a}], \qquad \text{for every } i = n, \dots, 0. \tag{36}$$

Indeed, if v_i is a leaf in $\mathfrak{T}_\sigma$, then either $\sigma_i = \top$ or $\sigma_i[\mathfrak{a}] = \mathfrak{a}(p) = a_i$ for some p. Now suppose inductively that (36) holds for every ℓ with $i < \ell \le n$. We have $a_i \le \Diamond_R\, a_\ell$ for every v_ℓ with $(v_i, v_\ell) \in R_\sigma$. So, by IH and monotonicity, we have

$$a_i \le a_i \wedge \bigwedge_{(v_i, v_\ell) \in R_\sigma} \Diamond_R\, \sigma_\ell[\mathfrak{a}] \le \sigma_i[\mathfrak{a}],$$

as required in (36). In particular, $a_0 \le \sigma_0[\mathfrak{a}] = \sigma[\mathfrak{a}]$. As $\mathfrak{A} \models (\sigma \to \tau)[\mathfrak{a}]$,

$$a_0 \le \tau[\mathfrak{a}]. \tag{37}$$

Finally, to prove (b) for $j = 0$, suppose that $R_\tau(u_0, u_j)$ for some j. Then $\Diamond_R\, \tau_j$ is a conjunct of τ, therefore $\tau[\mathfrak{a}] \le \Diamond_R\, \tau_j[\mathfrak{a}]$, and so $b_0 = a_0 \le \Diamond_R\, \tau_j[\mathfrak{a}] \le \Diamond_R\, b_j$ by (37), (35) and monotonicity, thus establishing $(b_0, b_j) \in R^{\mathfrak{F}}$. ⊣

Theorem 35 has the following consequence about the spi-fragments of modal *grammar logics* [30]:

Corollary 36. *Every spi-logic axiomatised by sp-implications of the form*

$$\Diamond_{R_1} \dots \Diamond_{R_n} p \to \Diamond_{S_1} \dots \Diamond_{S_m} p, \qquad \textit{for } n \ge 0,\ m > 0,$$

is complex, and so complete.

In particular, the spi-logics $\mathsf{SPi}+\{\boldsymbol{\iota}_{dense}\}$ with $\boldsymbol{\iota}_{dense} = (\Diamond p \to \Diamond\Diamond p)$ (defining *density*) and $\mathsf{SPi} + \{\Diamond_R \Diamond_S p \to \Diamond_S \Diamond_R p,\ \Diamond_S \Diamond_R p \to \Diamond_R \Diamond_S p\}$ (defining *commutativity*) are complex and complete. On the other hand, Corollary 36 gives examples of complete but undecidable finitely axiomatisable spi-logics [75, 66, 20, 2, 11], which clearly cannot have the finite frame property.

The following theorem will be used in §8.

Theorem 37. *Suppose R, S and Z are distinct elements in some signature $\mathcal{R}$, and let Σ consist of the following sp-implications: $\boldsymbol{\iota}_{fun}$ for $\Diamond_R$, $\boldsymbol{\iota}_{fun}$ for $\Diamond_S$,*

$$\Diamond_R \Diamond_S p \to \Diamond_S \Diamond_R p \quad \textit{and} \quad \Diamond_S \Diamond_R p \to \Diamond_R \Diamond_S p, \tag{38}$$

$$\Diamond_Z \top \to p, \tag{39}$$

$$\Diamond_X \Diamond_Z \top \to \Diamond_Z \top, \quad \textit{for all } X \in \mathcal{R}. \tag{40}$$

Then the spi-logic $\mathsf{SPi} + \Sigma$ is complex.

Proof. Let $\mathfrak{A} = (A, \wedge, \top, \Diamond_X)_{X\in\mathcal{R}}$ be a SLO such that $\mathfrak{A} \models \Sigma$. Then by (39), $\Diamond_Z \top$ is the $\leq$-smallest element in A. We call a filter U of $\mathfrak{A}$ *proper* if $\Diamond_Z \top \notin U$. As shown in the proof of Theorem 29 (i), for $X \in \{R, S\}$ and any filter U of $\mathfrak{A}$, $\Diamond_X^{-1}[U]$ is either empty or a filter. By (40), $\Diamond_X^{-1}[U]$ is either empty or a proper filter whenever U is proper. Let $\mathcal{F}_p(\mathfrak{A})$ be the set of all proper filters of $\mathfrak{A}$. We set, for $U, V \in \mathcal{F}_p(\mathfrak{A})$,

$$(U,V) \in X^{\mathfrak{G}} \iff V = \Diamond_X^{-1}[U], \quad \text{for } X \in \{R,S\},$$
$$(U,V) \in X^{\mathfrak{G}} \iff \Diamond_X[V] \subseteq U, \quad \text{for } X \in \mathcal{R} \setminus \{R,S,Z\}.$$

Then $R^{\mathfrak{G}}$ and $S^{\mathfrak{G}}$ are functional. Moreover, every $X \in \mathcal{R} \setminus \{Z\}$ satisfies (15) and (16) as well (with respect to $\mathcal{F}_p(\mathfrak{A})$), and so the map $f(a) = \{U \in \mathcal{F}_p(\mathfrak{A}) \mid a \in U\}$, for $a \in A$, embeds $\mathfrak{A}$ into $\mathfrak{G}^\star$ for $\mathfrak{G} = (\mathcal{F}_p(\mathfrak{A}), R^{\mathfrak{G}}, S^{\mathfrak{G}}, \emptyset, X^{\mathfrak{G}})_{X\in\mathcal{R}\setminus\{R,S,Z\}}$. Clearly, $\mathfrak{G}$ validates (39) and (40). It remains to show that $\mathfrak{G}$ validates (38), that is, $R^{\mathfrak{G}}$ and $S^{\mathfrak{G}}$ commute. Suppose that, say, $(U,V) \in R^{\mathfrak{G}}$ and $(V,W) \in S^{\mathfrak{G}}$, and let $Y = \Diamond_S^{-1}[U]$. We claim that $Y \neq \emptyset$, $(U,Y) \in S^{\mathfrak{G}}$ and $(Y,W) \in R^{\mathfrak{G}}$. Indeed, take some $a \in W$. Then $\Diamond_S a \in V$, and so $\Diamond_R \Diamond_S a \in U$. By (38), $\Diamond_S \Diamond_R a \in U$, and so $\Diamond_R a \in Y$, whence $Y \neq \emptyset$ and $a \in \Diamond_R^{-1}[Y]$. Therefore, $(U,Y) \in S^{\mathfrak{G}}$ and $W \subseteq \Diamond_R^{-1}[Y]$. The inclusion $\Diamond_R^{-1}[Y] \subseteq W$ is similar, proving $(Y,W) \in R^{\mathfrak{G}}$. The other direction of (38) is shown analogously. $\dashv$

§7. Completeness of spi-logics with disjunctive correspondents. Finally, we consider sp-implications whose correspondents contain disjunction, starting with a simple example.

Example 38. The spi-logic $\mathsf{SPi}+\{\boldsymbol{\iota}\}$ with $\boldsymbol{\iota} = (p \wedge \Diamond_R p \to \Diamond_S p)$ corresponding to the non-Horn, disjunctive first-order condition

$$\Psi_{\boldsymbol{\iota}} = \forall x, y\, \big(R(x,y) \to S(x,x) \vee S(x,y)\big) \tag{41}$$

is not complete. It is easy to see that $\{\boldsymbol{\iota}\} \models_{\mathsf{Kr}} p \wedge \Diamond_R \Diamond_S p \to \Diamond_S \Diamond_S p$. However, the SLO in Fig. 8 validates $\boldsymbol{\iota}$, but refutes $p \wedge \Diamond_R \Diamond_S p \to \Diamond_S \Diamond_S p$ when p is $\{1, 4\}$.

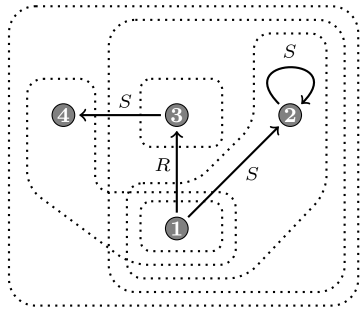


FIGURE 8. The SLO of Example 38.

7.1. Sp-implications defining n-functionality. Let $P = \{p_0, \dots, p_n\}$, for $n \geq 1$, and let

$$\boldsymbol{\iota}^n_{fun} \;=\; \big(\bigwedge_{Q \subseteq P,\, |Q|=n} \Diamond \bigwedge Q \;\to\; \Diamond \bigwedge P\big). \tag{42}$$

In particular, $\boldsymbol{\iota}^1_{fun} = \boldsymbol{\iota}_{fun}$. It is easy to see that $\boldsymbol{\iota}^n_{fun}$ corresponds to *n-functionality*:

$$\forall x, y_0, \dots, y_n \; \big(\bigwedge_{i \leq n} R(x, y_i) \to \bigvee_{i \neq j} (y_i = y_j)\big).$$

THEOREM 39. *None of* $\mathsf{SPi} + \{\boldsymbol{\iota}^n_{fun}\}$, $\mathsf{SPi} + \{\boldsymbol{\iota}_{refl}, \boldsymbol{\iota}^n_{fun}\}$, $\mathsf{SPi} + \{\boldsymbol{\iota}_{trans}, \boldsymbol{\iota}^n_{fun}\}$, $\mathsf{SPi} + (\Sigma_{qo} \cup \{\boldsymbol{\iota}^n_{fun}\})$, *and* $\mathsf{SPi} + (\Sigma_{equiv} \cup \{\boldsymbol{\iota}^n_{fun}\})$ *is complex, for* $n \geq 2$.

PROOF. Let $\mathfrak{A}_n$ be the SLO in Fig. 9. It is easy to see that $\mathfrak{A}_n \models \Sigma_{equiv} \cup \{\boldsymbol{\iota}^n_{fun}\}$ if $n \geq 2$. Now suppose there is an sp-embedding $\eta : \mathfrak{A}_n \to \mathfrak{F}^\star$ for some frame $\mathfrak{F} = (W, R^{\mathfrak{F}})$. Then there is some $x \in W \setminus \eta(g)$. As $W = \eta(\Diamond a_i) = \Diamond^+ \eta(a_i)$ for all $i \leq n$, there exist $y_0 \in \eta(a_0)$, ..., $y_n \in \eta(a_n)$ such that $(x, y_i) \in R^{\mathfrak{F}}$ for all $i \leq n$. As $\eta(g) = \eta(\Diamond g) = \Diamond^+ \eta(g)$, we have $y_i \notin \eta(g)$, for any $i \leq n$. It follows that all the y_i are distinct, showing that $\mathfrak{F}$ is not n-functional. ⊣

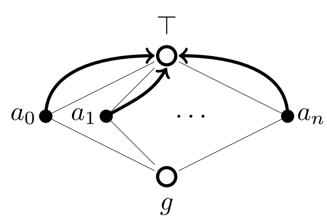


FIGURE 9. The SLO $\mathfrak{A}_n$ in the proof of Theorem 39.

THEOREM 40. $\mathsf{SPi} + \{\boldsymbol{\iota}^n_{fun}\}$ *is complete, for any* $n \geq 1$.

PROOF. For $n = 1$, this is Theorem 29 (i). For $n \geq 2$, we prove the theorem by the syntactic proxies method from §4.2. We first define $\{\boldsymbol{\iota}^n_{fun}\}$-*normal forms* by induction: (i) all propositional variables and $\top$ are $\{\boldsymbol{\iota}^n_{fun}\}$-normal forms; (ii) if $\tau_1, \dots, \tau_n$ are $\{\boldsymbol{\iota}^n_{fun}\}$-normal forms, then so is $\Diamond(\tau_1 \wedge \dots \wedge \tau_n)$.

CLAIM 40.1. *For any sp-formula ϱ, there is a set N_ϱ of $\{\boldsymbol{\iota}^n_{fun}\}$-normal forms such that*

$$\{\boldsymbol{\iota}^n_{fun}\} \vdash_{\mathsf{SLO}} \varrho \approx \bigwedge N_\varrho.$$

PROOF. The proof is by induction on the modal depth d of ϱ. The basis $d = 0$ is trivial. Suppose inductively that ϱ is an sp-formula of depth $d > 0$. Then $\varrho = \bigwedge P_\varrho \wedge \Diamond\varrho_1 \wedge \dots \wedge \Diamond\varrho_k$, where P_ϱ is a set consisting of propositional variables and $\top$, and each ϱ_i is an sp-formula of depth $\leq d-1$. By IH, $\{\boldsymbol{\iota}^n_{fun}\} \vdash_{\mathsf{SLO}} \varrho_i \approx \bigwedge A_i$, for some set A_i of $\{\boldsymbol{\iota}^n_{fun}\}$-normal forms and $i = 1, \dots, k$. Then

$$\{\boldsymbol{\iota}^n_{fun}\} \vdash_{\mathsf{SLO}} \varrho \approx (\bigwedge P_\varrho \wedge \bigwedge_{i=1}^{k} \Diamond \bigwedge A_i).$$

If $|A_i| \leq n$ for all i, then we are done. So fix some i and suppose $|A_i| = k > n$. Then we always have $\vdash_{\mathsf{SLO}} \Diamond \bigwedge A_i \to \bigwedge_{Q \subseteq A_i,\, |Q|=n} \Diamond \bigwedge Q$. We show that

$$\{\boldsymbol{\iota}^n_{fun}\} \vdash_{\mathsf{SLO}} \bigwedge_{\substack{Q \subseteq A_i,\\ |Q|=n}} \Diamond \bigwedge Q \to \Diamond \bigwedge A_i. \tag{43}$$

In order to prove this, first we claim that $\{\boldsymbol{\iota}^m_{fun}\} \vdash_{\mathsf{SLO}} \boldsymbol{\iota}^{m+1}_{fun}$, for every m. Indeed,

$$\{\boldsymbol{\iota}^m_{fun}\} \vdash_{\mathsf{SLO}} \bigwedge_{\substack{Q \subseteq \{p_0, \dots, p_{m+1}\}\\ |Q|=m+1}} \Diamond \bigwedge Q \to \bigwedge_{\substack{Q \subseteq \{p_1, \dots, p_{m+1}\}\\ |Q|=m}} \Diamond (p_0 \wedge \bigwedge Q) \to$$

$$\bigwedge_{\substack{Q \subseteq \{p_1, \dots, p_{m+1}\}\\ |Q|=m}} \Diamond \bigwedge_{q \in Q} (p_0 \wedge q) \to \Diamond \bigwedge_{q \in \{p_1, \dots, p_{m+1}\}} (p_0 \wedge q) \approx \Diamond(p_0 \wedge \dots \wedge p_{m+1}).$$

Therefore, we have $\{\boldsymbol{\iota}^n_{fun}\} \vdash_{\mathsf{SLO}} \boldsymbol{\iota}^m_{fun}$, for every $m > n$. Thus,

$$\{\boldsymbol{\iota}^n_{fun}\} \vdash_{\mathsf{SLO}} \bigwedge_{\substack{Q \subseteq \{p_0, \dots, p_{k-1}\}\\ |Q|=n}} \Diamond \bigwedge Q \;\to \bigwedge_{\substack{Q \subseteq \{p_0, \dots, p_{k-1}\}\\ |Q|=n+1}} \Diamond \bigwedge Q \;\to\; \dots$$

$$\dots \;\to\; \Diamond(p_0 \wedge \dots \wedge p_{k-1}),$$

and so a substitution of the k terms in A_i for $p_0, \dots, p_{k-1}$ in $\boldsymbol{\iota}^k_{fun}$ gives (43). $\dashv$

CLAIM 40.2. *For any sp-formula σ and $\{\boldsymbol{\iota}^n_{fun}\}$-normal form τ, if $\{\boldsymbol{\iota}^n_{fun}\} \models_{\mathsf{Kr}} \sigma \to \tau$ then $\models_{\mathsf{Kr}} \sigma \to \tau$.*

PROOF. The proof is by induction on the modal depth d of τ. The basis is again easy. Now assume inductively that the claim holds for d and the depth of τ is $d+1$. Let $\sigma = \bigwedge P_\sigma \wedge \Diamond\sigma_1 \wedge \ldots \wedge \Diamond\sigma_k$, where P_σ is some set of propositional variables and $\top$, and each σ_i is an sp-formula. Suppose $\tau = \Diamond(\tau_1 \wedge \ldots \wedge \tau_n)$, where each τ_i is either a variable, or $\top$, or of the form $\Diamond(\tau^i_1 \wedge \dots \wedge \tau^i_n)$. Let $\not\models_{\mathsf{Kr}} \sigma \to \tau$. Then, for every j ($1 \leq j \leq k$), there is i ($1 \leq i \leq n$) such that $\not\models_{\mathsf{Kr}} \sigma_j \to \tau_i$, and so $\bigcup_{i=1}^n K_i = \{1, \dots, k\}$, for $K_i = \{1 \leq j \leq k \mid \not\models_{\mathsf{SLO}} \sigma_j \to \tau_i\}$. It is not hard to see that, for any i with $K_i \neq \emptyset$, we have $\not\models_{\mathsf{Kr}} (\bigwedge_{j \in K_i} \sigma_j) \to \tau_i$. By IH, for any such i, there is a Kripke model $\mathfrak{M}_i$ based on an n-functional frame with root r_i where $\bigwedge_{j \in K_i} \sigma_j$ holds, but τ_i does not. Now take a fresh node r, make $\bigwedge P_\sigma$ true in r, and connect r to r_i of each $\mathfrak{M}_i$. The constructed Kripke model is based on an n-functional frame and refutes $\sigma \to \tau$ at r, showing that $\{\boldsymbol{\iota}^n_{fun}\} \not\models_{\mathsf{Kr}} \sigma \to \tau$ as required. ⊣

That $\mathsf{SPi}+\{\boldsymbol{\iota}^n_{fun}\}$ is complete follows now from Claims 40.1, 40.2, completeness of SPi (Theorem 4) and (7). ⊣

Now, we set

$$\Sigma^n_{equiv} \;=\; \Sigma_{equiv} \cup \{\boldsymbol{\iota}^n_{fun}\}, \qquad \text{for } 1 \le n < \omega.$$

and $\mathsf{SPi}^n_{equiv} = \mathsf{SPi} + \Sigma^n_{equiv}$. The correspondent Φ_n of Σ^n_{equiv} says that R is an equivalence relation whose classes (clusters) are of size $\le n$.

THEOREM 41. ([46])[8] SPi^n_{equiv} *is complete, for every* $n \ge 2$.

PROOF. The proof is again by the method of syntactic proxies from §4.2. Now, Σ^n_{equiv}-*normal forms* are defined as propositional variables, $\top$ and sp-formulas of the form $\Diamond(q_1 \wedge \cdots \wedge q_n)$, where the q_i are propositional variables or $\top$.

CLAIM 41.1. *For any sp-formula ϱ, there is a set N_ϱ of Σ^n_{equiv}-normal forms with*

$$\Sigma^n_{equiv} \vdash_{\mathsf{SLO}} \varrho \approx \bigwedge N_\varrho.$$

PROOF. As $\Sigma_{equiv} \vdash_{\mathsf{SLO}} \Diamond p \wedge \Diamond q \approx \Diamond(p \wedge \Diamond q)$ (by $\mathsf{SPi}_{equiv} = \mathsf{SPi}'_{equiv}$ and $\boldsymbol{\iota}_{eucl}$), it is easy to see that, for any $\{\boldsymbol{\iota}^n_{fun}\}$-normal form α (as defined in the proof of Theorem 40), there is some Σ^n_{equiv}-normal form β such that $\Sigma_{equiv} \vdash_{\mathsf{SLO}} \alpha \approx \beta$. Now the claim follows from Claim 40.1. In particular,

$$N_\varrho = P_{r_\varrho} \cup \{\Diamond \bigwedge Q \mid Q \subseteq P_x,\ |Q| \le n,\ x \text{ in } \mathfrak{M}_\varrho\},$$

where P_x is the set of variables that are true at x in the ϱ-tree model $\mathfrak{M}_\varrho$. ⊣

CLAIM 41.2. *For any sp-formula σ and Σ^n_{equiv}-normal form τ, if $\Sigma^n_{equiv} \models_{\mathsf{Kr}} \sigma \to \tau$ then $\Sigma_{equiv} \models_{\mathsf{Kr}} \sigma \to \tau$.*

PROOF. Suppose $\Sigma_{equiv} \not\models_{\mathsf{Kr}} \sigma \to \tau$. Let $R^\forall_\sigma = W_\sigma \times W_\sigma$ for the domain W_σ of the σ-tree model $\mathfrak{M}_\sigma$. Consider the Kripke model $\mathfrak{M}^\forall_\sigma = (\mathfrak{T}^\forall_\sigma, \mathfrak{v}_\sigma)$ over $\mathfrak{T}^\forall_\sigma = (W_\sigma, R^\forall_\sigma)$. As $\mathfrak{M}^\forall_\sigma$ is the equivalence-closure of $\mathfrak{M}_\sigma$, we have $\mathfrak{M}^\forall_\sigma, r_\sigma \models \sigma$ and $\mathfrak{M}^\forall_\sigma, r_\sigma \not\models \tau$ by Proposition 12, and so $\tau \ne \top$. If τ is a propositional variable p, then take the following model $\mathfrak{M}$ based on the universal frame over $\{x, y\}$: for each variable q, let $\mathfrak{M}, x \models q$ iff $r_\sigma \in \mathfrak{v}_\sigma(q)$ and $\mathfrak{M}, y \models q$ iff $\mathfrak{v}_\sigma(q) \setminus \{r_\sigma\} \ne \emptyset$. (That is, $\mathfrak{M}$ is obtained from $\mathfrak{M}^\forall_\sigma$ by 'sticking together' all of its points different from r_σ.) Then we clearly have $\mathfrak{M}, x \not\models (\sigma \to p)$. Finally, let τ be of the form $\Diamond(q_1 \wedge \cdots \wedge q_n)$. If W_σ contains $\le n$ points, then $\Sigma^n_{equiv} \not\models_{\mathsf{Kr}} \sigma \to \tau$. So suppose $W_\sigma = \{w_1, \dots, w_m\}$ for some $m > n$. We show that there is a Kripke model $\mathfrak{M}$ based on a universal frame with $< m$ points and such that $\mathfrak{M} \not\models (\sigma \to \tau)$. Indeed, as $\mathfrak{M}^\forall_\sigma, r_\sigma \not\models \Diamond(q_1 \wedge \cdots \wedge q_n)$, for every $1 \le i \le m$ there is $Q_i \subseteq \{q_1, \dots, q_n\}$ such that $|Q_i| = n-1$ and $\{q_k \mid 1 \le k \le n,\ \mathfrak{M}^\forall_\sigma, w_i \models q_k\} \subseteq Q_i$. So by the pigeonhole principle, there are $i \ne j$ with $Q_i = Q_j$. Now let $\mathfrak{M}$ result from $\mathfrak{M}^\forall_\sigma$

[8] This result also follows from [46], which showed (for the similarity type without $\top$) that $\mathsf{SPi}^n_{equiv} \models_{\mathsf{BAO}}$ is conservative over $\mathsf{SPi}^n_{equiv} \models_{\mathsf{SLO}}$.

by 'sticking together' w_i and w_j. Then we have $\mathfrak{M}, r_\sigma \not\models \sigma \to \Diamond(q_1 \wedge \cdots \wedge q_n)$, and so $\mathfrak{M} \not\models (\sigma \to \tau)$, as required. ⊣

That SPi^n_{equiv} is complete follows now from Claims 41.1, 41.2, completeness of SPi_{equiv} (Theorem 23 (*ii*)) and (7). ⊣

As a consequence of Claims 41.1 and 41.2 we also obtain:

THEOREM 42. SPi^n_{equiv} *is decidable in* PTIME*, for every* $n \geq 2$.

PROOF. Follows from the tractability of SPi_{equiv} (Theorem 13) and the fact that $|N_\varrho|$ in Claim 41.1 is clearly polynomial in the size of $\mathfrak{M}_\varrho$. ⊣

Jackson [46] also proves the following about extensions of SPi_{equiv}:

THEOREM 43. ([46]) *Let* L *be any non-trivial spi-logic extending* SPi_{equiv}. *Then exactly one of the following cases holds:*

- $L = \mathsf{SPi}_{equiv}$,
- $L = \mathsf{SPi} + (\Sigma_{equiv} \cup \{\Diamond p \to p\})$,
- $L = \mathsf{SPi} + (\Sigma_{equiv} \cup \{\Diamond p \to \Diamond q\})$,
- $L = \mathsf{SPi}^n_{equiv}$, *for some* n $(1 \leq n < \omega)$.

Then Proposition 32, Theorems 23 (*ii*), 28, 29 (*iv*), 31 and 43 give a full classification of the extensions SPi_{equiv} according to their completeness: the trivial spi-logic, SPi_{equiv}, $\mathsf{SPi} + (\Sigma_{equiv} \cup \{\Diamond p \to p\})$ and SPi^n_{equiv}, for $1 < n < \omega$, are complete, while SPi^1_{equiv} and $\mathsf{SPi} + (\Sigma_{equiv} \cup \{\Diamond p \to \Diamond q\})$ are incomplete.

By Theorem 28, $\mathsf{SPi} + (\Sigma_{equiv} \cup \{\Diamond p \to p\})$ is complex. Theorems 31, 39 and 43 imply that it is the only complex non-trivial proper extension of SPi_{equiv}:

COROLLARY 44. *Let* L *be any non-trivial spi-logic such that* $L \supseteq \mathsf{SPi}_{equiv}$, $L \neq \mathsf{SPi}_{equiv}$ *and* $L \neq \mathsf{SPi} + (\Sigma_{equiv} \cup \{\Diamond p \to p\})$. *Then* L *is not complex.*

Finally, we show that $\boldsymbol{\iota}^n_{fun}$ behaves differently when added to SPi_{qo}. A transitive frame $\mathfrak{F}$ is said to be *of depth* n, for $n \geq 1$, if $\mathfrak{F}$ contains a chain of n points from distinct clusters but no longer chain of this sort. It is easy to see that, over SPi_{qo}, we can define the property '$\mathfrak{F}$ is of depth $\leq n$' by the sp-implication

$$\boldsymbol{\iota}^n_{depth} = \big(p \wedge \underbrace{\Diamond\big(q \wedge \Diamond(p \wedge \dots) \dots\big)}_{\Diamond \text{ is used } n \text{ times}} \;\to\; \underbrace{\Diamond\big(q \wedge \Diamond(p \wedge \dots) \dots\big)}_{\Diamond \text{ is used } n+1 \text{ times}}\big).$$

Then $\boldsymbol{\iota}^1_{depth} = \boldsymbol{\iota}_{sym}$ and $\boldsymbol{\iota}^2_{depth}$ has the following 'disjunctive' correspondent:

$$\forall x, y, z\, \big(R(x,y) \wedge R(y,z) \to R(y,x) \vee R(z,y)\big).$$

Also, it is not hard to see that $\{\boldsymbol{\iota}^n_{depth}\} \models_{\mathsf{SLO}} \boldsymbol{\iota}^{n+1}_{depth}$, for all $n \geq 1$ (simply substitute $p \wedge \Diamond q$ for p, and $q \wedge \Diamond p$ for q in $\boldsymbol{\iota}^n_{depth}$).

QUESTION 3. *Are* $\mathsf{SPi}+\{\boldsymbol{\iota}^n_{depth}\}$, $\mathsf{SPi}+\{\boldsymbol{\iota}_{trans}, \boldsymbol{\iota}^n_{depth}\}$ *and* $\mathsf{SPi}+(\Sigma_{qo} \cup \{\boldsymbol{\iota}^n_{depth}\})$ *complete?*

As an n-functional reflexive and transitive frame can have at most n points, its depth must be $\leq n$. Therefore, for any $n \geq 1$, we have

$$\Sigma_{qo} \cup \{\boldsymbol{\iota}^n_{fun}\} \models_{\mathsf{Kr}} \boldsymbol{\iota}^n_{depth}. \tag{44}$$

THEOREM 45. $\Sigma_{qo} \cup \{\boldsymbol{\iota}^n_{fun}\} \not\models_{\mathsf{SLO}} \boldsymbol{\iota}^n_{depth}$, *for any* $n \geq 2$.

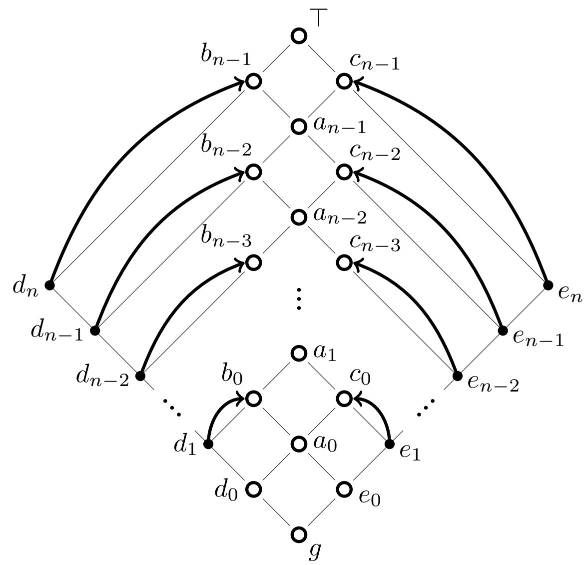


FIGURE 10. The SLO $\mathfrak{A}_n$ in the proof of Theorem 45.

PROOF. Fix some $n \geq 2$ and take the SLO $\mathfrak{A}_n$ in Fig. 10. It is easy to check that $\mathfrak{A}_n \models \Sigma_{qo}$. We claim that $\mathfrak{A}_n \models \boldsymbol{\iota}^n_{fun}$. In fact, if $n \geq 3$ then $\mathfrak{A}_n \models \boldsymbol{\iota}^3_{fun}$. We prove only the latter. Take any valuation $\mathfrak{a}$ in $\mathfrak{A}_n$. If $\mathfrak{a}(p_0), \dots, \mathfrak{a}(p_3)$ are not pairwise $\leq$-incomparable, then $\mathfrak{A}_n \models \boldsymbol{\iota}^3_{fun}[\mathfrak{a}]$ clearly holds. And if they are, then $d_i, e_j \in \{\mathfrak{a}(p_0), \dots, \mathfrak{a}(p_3)\}$ must hold for some $i, j \leq n$. As $d_i \wedge e_j = g$, the left-hand side of $\boldsymbol{\iota}^3_{fun}[\mathfrak{a}]$ evaluates to g.

On the other hand, we claim that $\mathfrak{A}_n \not\models \boldsymbol{\iota}^n_{depth}[\mathfrak{a}]$ for the valuation $\mathfrak{a}(p) = d_n$ and $\mathfrak{a}(q) = e_n$. Indeed, for $1 \leq k \leq n$, define sp-formulas τ_k and σ_k by taking $\tau_1 = \Diamond q$, $\sigma_1 = \Diamond p$, $\tau_k = \Diamond(q \wedge \sigma_{k-1})$ and $\sigma_k = \Diamond(p \wedge \tau_{k-1})$. Then $\boldsymbol{\iota}^n_{depth}$ is $p \wedge \tau_n \to \tau_{n+1}$. It is not hard to prove by parallel induction that $\tau_k[\mathfrak{a}] = c_{n-k}$ and $\sigma_k[\mathfrak{a}] = b_{n-k}$ for all $1 \leq k \leq n$. Therefore, the left-hand side of $\boldsymbol{\iota}^n_{depth}$ evaluates to $d_n \wedge c_0 = d_0$, while the right-hand side to $\Diamond(e_n \wedge b_0) = \Diamond e_0 = e_0$. $\dashv$

As a consequence of (44), Theorems 29 and 45 we obtain:

COROLLARY 46. $\mathsf{SPi} + (\Sigma_{qo} \cup \{\boldsymbol{\iota}^n_{fun}\})$ *is not complete, for any* $n \geq 1$.

7.2. Sp-implications defining width above SPi_{qo}. Consider the sp-implication

$$\boldsymbol{\iota}_{wcon} = \big(\Diamond(p \wedge q) \wedge \Diamond(p \wedge r) \to \Diamond(p \wedge \Diamond q \wedge \Diamond r)\big),$$

with the disjunctive correspondent

$$\forall x, y, z\, \big(R(x,y) \wedge R(x,z) \to \big(R(y,y) \wedge R(y,z)\big) \vee \big(R(z,z) \wedge R(z,y)\big)\big). \tag{45}$$

Now let

$$\Sigma_{lin} = \{\boldsymbol{\iota}_{refl}, \boldsymbol{\iota}_{trans}, \boldsymbol{\iota}_{wcon}\}. \tag{46}$$

It is easy to see that Σ_{lin} defines the class of all linear quasiorders (frames for the modal logic $\mathsf{S4.3}$). We set

$$\mathsf{SPi}_{lin} = \mathsf{SPi} + \Sigma_{lin}.$$

THEOREM 47. *Neither* $\mathsf{SPi} + \{\boldsymbol{\iota}_{wcon}\}$ *nor* SPi_{lin} *is complex.*

PROOF. Take the SLO $\mathfrak{A}$ in Fig. 11. It is not hard to check that $\mathfrak{A} \models \Sigma_{lin}$. Now

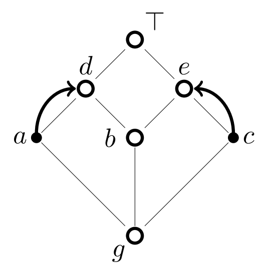


FIGURE 11. The SLO $\mathfrak{A}$ in the proof of Theorem 47.

suppose we have an sp-embedding $\eta : \mathfrak{A} \to \mathfrak{F}^\star$, for some $\mathfrak{F} = (W, R^{\mathfrak{F}})$. Then there is $u \in \eta(b) \setminus \eta(g)$. As $b \le \Diamond a$ and $b \le \Diamond c$, we have $\eta(b) \subseteq \eta(\Diamond a) = \Diamond^+\eta(a)$ and $\eta(b) \subseteq \eta(\Diamond c) = \Diamond^+\eta(c)$. Then $u \in \Diamond^+\eta(a) \cap \Diamond^+\eta(c)$, and so there are $v \in \eta(a)$ and $w \in \eta(c)$ such that $(u, v) \in R^{\mathfrak{F}}$ and $(u, w) \in R^{\mathfrak{F}}$. As $\eta(g) = \eta(\Diamond g) = \Diamond^+\eta(g)$, we have $v \notin \eta(g) = \eta(a \wedge \Diamond c) = \eta(a) \cap \Diamond^+\eta(c)$, and so $(v, w) \notin R^{\mathfrak{F}}$. Similarly, $w \notin \eta(g) = \eta(c \wedge \Diamond a) = \eta(c) \cap \Diamond^+\eta(a)$, and so $(w, v) \notin R^{\mathfrak{F}}$. Therefore, (45) does not hold in $\mathfrak{F}$. ⊣

Now we use the syntactic proxies method to prove the following:

THEOREM 48. SPi_{lin} *is complete.*

PROOF. We define Σ_{lin}-*normal forms* by induction: (i) all finite conjunctions of propositional variables are Σ_{lin}-normal forms; (ii) if τ is an Σ_{lin}-normal form and P_τ is a set of propositional variables, then $\bigwedge P_\tau \wedge \Diamond\tau$ is an Σ_{lin}-normal form.

CLAIM 48.1. *For any sp-formula* ϱ*, there is a set* N_ϱ *of* Σ_{lin}*-normal forms with*

$$\Sigma_{lin} \vdash_{\mathsf{SLO}} \big(\varrho \approx \bigwedge N_\varrho\big).$$

PROOF. Let N_ϱ be the set of normal forms describing the full linear branches of $\mathfrak{M}_\varrho$ (from root to a leaf): if $\varrho = \bigwedge P_\varrho$ then $N_\varrho = \{\varrho\}$, and if $\varrho = \bigwedge P_\varrho \wedge \bigwedge_{i<k} \Diamond\varrho_i$ then $N_\varrho = \{\bigwedge P_\varrho \wedge \Diamond\tau \mid \tau \in \bigcup_{i<k} N_{\varrho_i}\}$. We clearly have $\vdash_{\mathsf{SLO}} \big(\varrho \to \bigwedge N_\varrho\big)$. Conversely, it is easy to see first that, for any n,

$$\Sigma_{lin} \vdash_{\mathsf{SLO}} \Diamond(p \wedge \Diamond q_1) \wedge \dots \wedge \Diamond(p \wedge \Diamond q_n) \to \Diamond(p \wedge \Diamond q_1 \wedge \dots \wedge \Diamond q_n). \tag{47}$$

Next, we prove by induction on ϱ that

$$\Sigma_{lin} \vdash_{\mathsf{SLO}} \bigwedge_{\tau \in N_\varrho} \Diamond\tau \ \to\ \Diamond\varrho. \tag{48}$$

This is obvious if the depth of ϱ is 0. So suppose $\varrho = \bigwedge P_\varrho \wedge \bigwedge_{i<k} \Diamond\varrho_i$. Then

$$\begin{aligned}
\Sigma_{lin} \vdash_{\mathsf{SLO}} \bigwedge_{\tau\in N_\varrho} \Diamond\tau \approx \bigwedge_{i<k} \bigwedge_{\tau\in N_{\varrho_i}} \Diamond(\bigwedge P_\varrho \wedge \Diamond\tau) \ &\to \quad \text{(by (47))}\\
&\to \ \bigwedge_{i<k} \Diamond(\bigwedge P_\varrho \wedge \bigwedge_{\tau\in N_{\varrho_i}} \Diamond\tau) \quad \text{(by IH)}\\
&\to \ \bigwedge_{i<k} \Diamond(\bigwedge P_\varrho \wedge \Diamond\varrho_i) \quad \text{(by (47))}\\
&\to \ \Diamond(\bigwedge P_\varrho \wedge \bigwedge_{i<k} \Diamond\varrho_i) \ \approx \ \Diamond\varrho.
\end{aligned}$$

Finally, by (48), for every $i < k$,

$$\Sigma_{lin} \vdash_{\mathsf{SLO}} \bigwedge N_\varrho \ \to \ \bigwedge_{\tau\in N_{\varrho_i}} \Diamond\tau \ \to \ \Diamond\varrho_i.$$

Since $\Sigma_{lin} \vdash_{\mathsf{SLO}} \bigwedge N_\varrho \to \bigwedge P_\varrho$, we have $\Sigma_{lin} \vdash_{\mathsf{SLO}} \bigwedge N_\varrho \to \varrho$ as required. $\dashv$

CLAIM 48.2. *For any sp-formula σ and any Σ_{lin}-normal form τ, $\Sigma_{lin} \models_{\mathsf{Kr}} \sigma \to \tau$ implies $\Sigma_{qo} \models_{\mathsf{Kr}} \sigma \to \tau$.*

PROOF. Suppose $\Sigma_{qo} \not\models_{\mathsf{Kr}} \sigma \to \tau$. Take the σ-tree model $\mathfrak{M}_\sigma$, and let $\mathfrak{M}^*_\sigma = \big((W_\sigma, R^*_\sigma), \mathfrak{v}_\sigma\big)$ for the reflexive and transitive closure R^*_σ of R_σ. By Proposition 12, we have $\mathfrak{M}^*_\sigma, r_\sigma \not\models \tau$. We call $\mathfrak{M} = (W_\sigma, R, \mathfrak{v}_\sigma)$ a *linearisation* of $\mathfrak{M}^*_\sigma$ if R is a linear order[9] containing R^*_σ.

It should be clear that $\mathfrak{M}, r_\sigma \models \sigma$ for any linearisation $\mathfrak{M}$ of $\mathfrak{M}^*_\sigma$. We show that there is a linearisation $\mathfrak{M}^+_\sigma$ of $\mathfrak{M}^*_\sigma$ with $\mathfrak{M}^+_\sigma, r_\sigma \not\models \tau$, which means that $\mathsf{SPi}_{lin} \not\models_{\mathsf{Kr}} \sigma \to \tau$. We construct $\mathfrak{M}^+_\sigma$ step-by-step by rearranging the points in W_σ. We build a binary tree $(\mathbb{L}_\sigma, \prec)$ of models $\mathfrak{M} = (W_\sigma, R, \mathfrak{v}_\sigma)$ by induction so that each (W_σ, R) is a reflexive and transitive tree containing R^*_σ and, for each $\mathfrak{M}$ in $\mathbb{L}_\sigma$, there is some $\mathfrak{M}'$ with $\mathfrak{M} \prec \mathfrak{M}'$ and $\mathfrak{M}', r_\sigma \not\models \tau$. Each leaf in $(\mathbb{L}_\sigma, \prec)$ will be a linearisation of $\mathfrak{M}^*_\sigma$. First, let $\mathfrak{M}^*_\sigma$ be the root of $(\mathbb{L}_\sigma, \prec)$. Suppose now inductively that $\mathfrak{M} = (W_\sigma, R, \mathfrak{v}_\sigma)$ in $\mathbb{L}_\sigma$ has been defined, $\mathfrak{M}, r_\sigma \not\models \tau$, and R is not a linear order. We call a triple $(u, v_{\mathsf{left}}, v_{\mathsf{right}})$ of distinct points in W_σ an *R-defect* if $(u, v_{\mathsf{left}}) \in R$, $(u, v_{\mathsf{right}}) \in R$, $v_{\mathsf{left}} \neq v_{\mathsf{right}}$, but neither $(v_{\mathsf{left}}, v_{\mathsf{right}}) \in R$ nor $(v_{\mathsf{right}}, v_{\mathsf{left}}) \in R$ hold. Take any R-defect $(u, v_{\mathsf{left}}, v_{\mathsf{right}})$ with minimal R-distance between r_σ and u. We define two relations $R_{\mathsf{left}} = \big(R \setminus \{(u, v_{\mathsf{right}})\}\big) \cup \{(v_{\mathsf{left}}, v_{\mathsf{right}})\}$ and $R_{\mathsf{right}} = \big(R \setminus \{(u, v_{\mathsf{left}})\}\big) \cup \{(v_{\mathsf{right}}, v_{\mathsf{left}})\}$ (see Fig. 12), and add $\mathfrak{M} \prec \mathfrak{M}_{\mathsf{left}}$ and $\mathfrak{M} \prec \mathfrak{M}_{\mathsf{right}}$ to $(\mathbb{L}_\sigma, \prec)$, where $\mathfrak{M}_i = (W_\sigma, R_i, \mathfrak{v}_\sigma)$ for $i = \mathsf{left}, \mathsf{right}$.

We claim that either $\mathfrak{M}_{\mathsf{left}}, r_\sigma \not\models \tau$ or $\mathfrak{M}_{\mathsf{right}}, r_\sigma \not\models \tau$. Suppose otherwise. Then, by Proposition 7, there are two homomorphisms $h_{\mathsf{left}} \colon \mathfrak{M}_\tau \to \mathfrak{M}_{\mathsf{left}}$ and $h_{\mathsf{right}} \colon \mathfrak{M}_\tau \to \mathfrak{M}_{\mathsf{right}}$ with $h_{\mathsf{left}}(r_\tau) = h_{\mathsf{right}}(r_\tau) = r_\sigma$. If one of these is an $\mathfrak{M}_\tau \to \mathfrak{M}$ homomorphism, then $\mathfrak{M}, r_\sigma \models \tau$, contrary to IH. If this is not the case, suppose that $\mathfrak{M}_\tau$ is based on an irreflexive and intransitive unary tree $x_0 < x_1 < \cdots < x_n$. It is not hard to see that

- there is $i_{\mathsf{left}} < n$ such that $\big(h_{\mathsf{left}}(x_i), v_{\mathsf{left}}\big) \in R_{\mathsf{left}}$ for every $i \leq i_{\mathsf{left}}$, and $\big(v_{\mathsf{right}}, h_{\mathsf{left}}(x_i)\big) \in R_{\mathsf{left}}$ for every $i \geq i_{\mathsf{left}} + 1$;

[9] A *linear order* is an antisymmetric linear quasiorder.

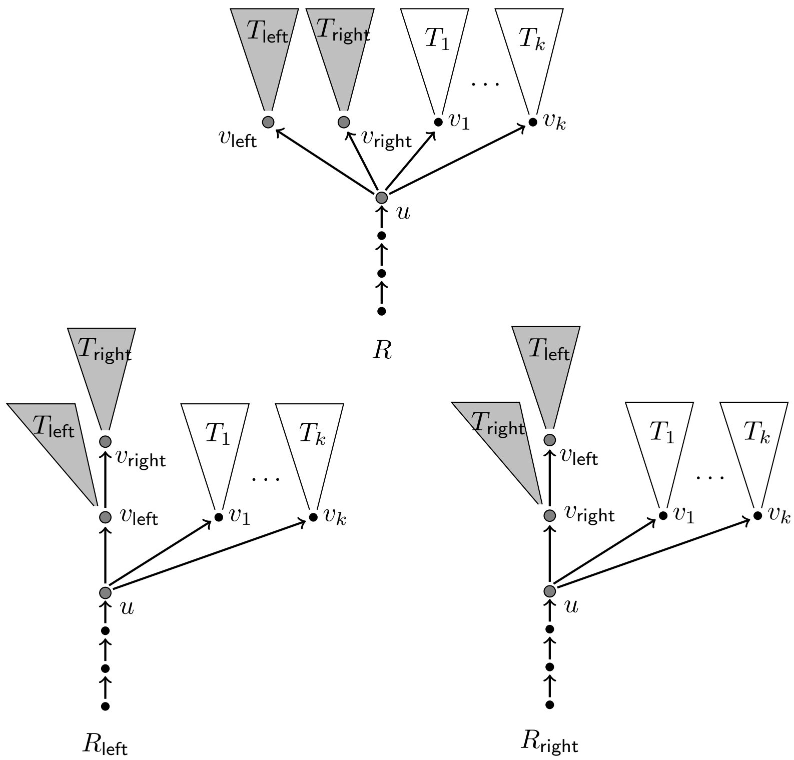


FIGURE 12. Linearising step-by-step.

– there is $i_{\mathsf{right}} < n$ such that $\big(h_{\mathsf{right}}(x_i), v_{\mathsf{right}}\big) \in R_{\mathsf{right}}$ for every $i \leq i_{\mathsf{right}}$, and $\big(v_{\mathsf{left}}, h_{\mathsf{right}}(x_i)\big) \in R_{\mathsf{right}}$ for every $i \geq i_{\mathsf{right}} + 1$.

Suppose $i_{\mathsf{right}} \geq i_{\mathsf{left}}$ (the other case is similar). Define h by taking, for any $i \leq n$,

$$h(x_i) = \begin{cases} h_{\mathsf{right}}(x_i), & \text{if } i \leq i_{\mathsf{right}}, \\ h_{\mathsf{left}}(x_i), & \text{else.} \end{cases}$$

We prove that h is an $\mathfrak{M}_\tau \to \mathfrak{M}$ homomorphism with $h(r_\tau) = r_\sigma$, from which we shall have $\mathfrak{M}, r_\sigma \models \tau$, contrary to IH. Thus, we need to show that, for every $i < n$, we have $\big(h(x_i), h(x_{i+1})\big) \in R$. There are three cases:

Case 1: $i < i_{\mathsf{right}}$. Then $h(x_i) = h_{\mathsf{right}}(x_i)$, $h_{\mathsf{right}}(x_{i+1}) = h(x_{i+1})$ and $\big(h(x_i), h(x_{i+1})\big) \in R_{\mathsf{right}}$. Since $i, i+1 \leq i_{\mathsf{right}}$, we have $\big(h(x_i), v_{\mathsf{right}}\big) \in R_{\mathsf{right}}$ and $\big(h(x_{i+1}), v_{\mathsf{right}}\big) \in R_{\mathsf{right}}$, and so $\big(h(x_i), h(x_{i+1})\big) \in R$ follows from $\big(h(x_i), h(x_{i+1})\big) \in R_{\mathsf{right}}$.

Case 2: $i > i_{\mathsf{right}}$. Then $h(x_i) = h_{\mathsf{left}}(x_i)$, $h_{\mathsf{left}}(x_{i+1}) = h(x_{i+1})$ and we also have $\big(h(x_i), h(x_{i+1})\big) \in R_{\mathsf{left}}$. As $i, i+1 \geq i_{\mathsf{left}} + 1$, we have $\big(v_{\mathsf{right}}, h(x_i)\big) \in R_{\mathsf{left}}$ and $\big(v_{\mathsf{right}}, h(x_{i+1})\big) \in R_{\mathsf{left}}$. Therefore, we obtain $\big(h(x_i), h(x_{i+1})\big) \in R$ from $\big(h(x_i), h(x_{i+1})\big) \in R_{\mathsf{left}}$.

Case 3: $i = i_{\mathsf{right}}$. Then $h(x_i) = h_{\mathsf{right}}(x_i)$, and so $\big(h(x_i), v_{\mathsf{right}}\big) \in R_{\mathsf{right}}$. Also, $h(x_{i+1}) = h_{\mathsf{left}}(x_{i+1})$ and, since $i+1 = i_{\mathsf{right}} + 1 \geq i_{\mathsf{left}} + 1$, we have $\big(v_{\mathsf{right}}, h(x_{i+1})\big) \in R_{\mathsf{left}}$. Therefore, $\big(h(x_i), h(x_{i+1})\big) \in R$, as required. ⊣

That SPi_{lin} is complete follows now from Claims 48.1, 48.2, completeness of SPi_{qo} (Corollary 16) and (7). ⊣

As a consequence of Claims 48.1 and 48.2 we also obtain:

THEOREM 49. SPi_{lin} *is decidable in* PTIME.

PROOF. Follows from the PTIME-time decidability of SPi_{qo} [71] (see also Theorem 13) and the fact that $|N_\varrho|$ in Claim 48.1 is the number of leaves in $\mathfrak{M}_\varrho$. ⊣

The completeness landscape for extensions of SPi_{lin} is much more involved than for extensions of SPi_{equiv}. In [51], all complete extensions of SPi_{lin} are characterised, and infinitely many incomplete extensions of $\mathsf{SPi} + (\Sigma_{lin} \cup \{\boldsymbol{\iota}^2_{fun}\})$ are given. Here we prove the following:

THEOREM 50. $\mathsf{SPi} + (\Sigma_{lin} \cup \{\boldsymbol{\iota}^n_{fun}\})$ *is not complete, for any* $n \geq 1$.

PROOF. For $n = 1$, we reuse the proof of Theorem 29 (iii) since we clearly have $\mathfrak{A} \models \boldsymbol{\iota}_{wcon}$. Now, fix some $n \geq 2$. Observe that $\mathsf{SPi}_{lin} = \mathsf{SPi} + (\Sigma_{qo} \cup \{\boldsymbol{\iota}'_{wcon}\})$, where

$$\boldsymbol{\iota}'_{wcon} = \big(\Diamond(p \wedge \Diamond q) \wedge \Diamond(p \wedge \Diamond r) \to \Diamond(p \wedge \Diamond q \wedge \Diamond r)\big). \tag{49}$$

Let $\mathfrak{A}_n$ be the SLO from the proof of Theorem 45. We claim that $\mathfrak{A}_n \models \boldsymbol{\iota}'_{wcon}$. Indeed, take a valuation $\mathfrak{a}$ in $\mathfrak{A}_n$. If there are distinct $x, y \in \{\mathfrak{a}(p), \Diamond q[\mathfrak{a}], \Diamond r[\mathfrak{a}]\}$ such that $x \leq y$, then $\mathfrak{A} \models \boldsymbol{\iota}'_{wcon}[\mathfrak{a}]$ clearly holds. So we may assume that $\mathfrak{a}(p)$, $\Diamond q[\mathfrak{a}]$, and $\Diamond r[\mathfrak{a}]$ are pairwise $\leq$-incomparable. Let, say, $\Diamond q[\mathfrak{a}] = b_i$, $\Diamond r[\mathfrak{a}] = c_i$, and $\mathfrak{a}(p) = d_j$, for some $i < n-1$ and $i+1 < j \leq n$ (the other cases are similar). Then both sides of $\boldsymbol{\iota}'_{wcon}$ evaluate to d_0 if $i = 0$, and to b_{i-1} if $i > 0$, proving that $\mathfrak{A} \models \boldsymbol{\iota}'_{wcon}[\mathfrak{a}]$. ⊣

QUESTION 4. *Is* $\mathsf{SPi} + (\Sigma_{lin} \cup \{\boldsymbol{\iota}^n_{depth}\})$ *complete for* $n > 1$*?*

The sp-implication $\boldsymbol{\iota}'_{wcon}$ in (49) was also used by Svyatlovsky [72] who showed that $\{\boldsymbol{\iota}_{trans}, \boldsymbol{\iota}'_{wcon}\}$ axiomatise the spi-fragment of $\mathsf{K4.3}$—the modal logic of all transitive and weakly connected frames—and described the class of Kripke frames validating $\{\boldsymbol{\iota}_{trans}, \boldsymbol{\iota}'_{wcon}\}$. As not all frames in this class are weakly connected, it follows that the class of $\mathsf{K4.3}$-frames is not spi-definable. For a direct model-theoretic proof of this fact, see Proposition 55 below. Svyatlovsky also proved that the spi-logic $\mathsf{SPi} + \{\boldsymbol{\iota}_{trans}, \boldsymbol{\iota}'_{wcon}\}$ is tractable.

We can generalise $\boldsymbol{\iota}_{wcon}$ to

$$\boldsymbol{\iota}^n_{width} = \Big(\Diamond\big(p \wedge \bigwedge P^n_0\big) \wedge \cdots \wedge \Diamond\big(p \wedge \bigwedge P^n_n\big) \to \Diamond(p \wedge \Diamond p_0 \wedge \cdots \wedge \Diamond p_n)\Big).$$

Then $\boldsymbol{\iota}^1_{wcon} = \boldsymbol{\iota}_{width}$.

QUESTION 5. *Are* $\mathsf{SPi} + (\Sigma_{qo} \cup \{\boldsymbol{\iota}^n_{width}\})$ *and* $\mathsf{SPi} + (\Sigma_{qo} \cup \{\boldsymbol{\iota}^m_{depth}, \boldsymbol{\iota}^n_{width}\})$ *complete?*

§8. Undecidability of completeness. Having established quite a few completeness and incompleteness results for spi-logics, we now show that an exhaustive and decidable classification of finitely axiomatisable spi-logics according to their completeness (or complexity) is not possible.

Theorem 51. *Given a finite set Σ of sp-implications, it is undecidable whether the spi-logic $\mathsf{SPi} + \Sigma$ is complete; it is also undecidable whether it is complex.*

Proof. We encode the halting problem for deterministic Turing machines starting from an empty tape. Recall that a Turing machine is a tuple

$$M = (Q, \Gamma, \delta, q_0, q_h),$$

where Q is a non-empty finite set of states with an initial state q_0 and a halting state q_h, Γ is a finite tape alphabet with a special symbol $b \in \Gamma$ denoting the blank cell, and δ is a transition function that, for any pair $(q, a) \in Q \times \Gamma$, gives a triple $\delta(q, a) \in Q \times \Gamma \times \{\mathsf{L}, \mathsf{R}\}$, where L and R stand for 'move left' and 'move right', respectively. We use the standard definition of a computation of M on an input word. Then the problem to decide whether the computation starting from an empty tape in state q_0 reaches the halting state q_h is undecidable [27]. We may assume that the initial state is not reachable from any state, the halting state has no successor state, and that the head never moves to the left of its initial position. Now, suppose $M = (Q, \Gamma, \delta, q_0, q_h)$ is such a Turing machine.

For the reduction, we encode the computation of M starting from the empty tape by a grid with points $d_{n,m}$ for the nth cell of the mth configuration of the computation. We use relations next and step such that $(d_{n,m}, d_{n+1,m}) \in \mathsf{next}$ and $(d_{n,m}, d_{n,m+1}) \in \mathsf{step}$. We encode that the nth cell contains symbol $a \in \Gamma$ in the mth configuration by introducing a relation R_a and stating that $d_{n,m}$ has some R_a-successor. Likewise, we encode that M is in state $q \in Q$ in the mth configuration by introducing a relation R_q and stating that all $d_{n,m}$ have an R_q-successor. In the same way, we use relations R_{head}, R_{left}, and R_{right} to encode the position of the head and, for technical reasons, all cells to the left and right of the position of the head.

Using the above intuition, we now construct a finite set Σ_M of sp-implications such that M halts on the empty tape iff $\mathsf{SPi} + \Sigma_M$ is complete iff $\mathsf{SPi} + \Sigma_M$ is complex. Let $\Diamond_{\mathsf{next}}$ and $\Diamond_{\mathsf{step}}$ be modal operators interpreted by the relations next and step introduced above. The following set of sp-implications state that the relations next and step are functional and commute:

(50) $\Diamond_{\mathsf{next}}\, p \land \Diamond_{\mathsf{next}}\, q \to \Diamond_{\mathsf{next}}\, (p \land q)$,

(51) $\Diamond_{\mathsf{step}}\, p \land \Diamond_{\mathsf{step}}\, q \to \Diamond_{\mathsf{step}}\, (p \land q)$,

(52) $\Diamond_{\mathsf{next}} \Diamond_{\mathsf{step}}\, p \to \Diamond_{\mathsf{step}} \Diamond_{\mathsf{next}}\, p$ and $\Diamond_{\mathsf{step}} \Diamond_{\mathsf{next}}\, p \to \Diamond_{\mathsf{next}} \Diamond_{\mathsf{step}}\, p$.

To axiomatise the properties of R_a, $a \in \Gamma$, R_q, $q \in Q$, and R_{head}, R_{left}, and R_{right}, we introduce an operator $\Diamond_q$ for every state $q \in Q$, an operator $\Diamond_a$ for every $a \in \Gamma$, and operators $\Diamond_{\mathsf{head}}$, $\Diamond_{\mathsf{left}}$ and $\Diamond_{\mathsf{right}}$. We say that M does not halt by the sp-implication

(53) $$\Diamond_{q_h} \top \to p.$$

In order to show that what we have so far axiomatise a complex spi-logic (see Theorem 37), we also need to add the sp-implications

$$\Diamond \Diamond_{q_h} \top \to \Diamond_{q_h} \top, \tag{54}$$

for all $\Diamond = \Diamond_{\mathsf{next}}, \Diamond_{\mathsf{step}}, \Diamond_{\mathsf{head}}, \Diamond_{\mathsf{left}}, \Diamond_{\mathsf{right}}, \Diamond_q, \Diamond_a, q \in Q, a \in \Gamma$. (Note that if the language contained a constant $\bot$, interpreted as 'falsehood' in Kripke models and the $\leq$-smallest element in 'normal' SLOs, then $\Diamond_{q_h} \top \to \bot$ would suffice in place of (53)–(54), see §9.2.) Let Ξ be the set of sp-implications comprising (50)–(54). By Theorem 37, the spi-logic $\mathsf{SPi} + \Xi$ is complex. To ensure that Ξ together with the set of sp-implications encoding the computation of M on empty tape axiomatise a complex spi-logic, we apply Proposition 5, and therefore represent states, tape symbols and tape positions using variable-free sp-formulas of the form $\Diamond_R \top$ for the operators $\Diamond_R$ introduced above. We first set left and right correctly, exploiting the assumed functionality of next:

$$\Diamond_{\mathsf{next}} \Diamond_{\mathsf{left}} \top \to \Diamond_{\mathsf{left}} \top, \tag{55}$$

$$\Diamond_{\mathsf{next}} \Diamond_{\mathsf{head}} \top \to \Diamond_{\mathsf{left}} \top, \tag{56}$$

$$\Diamond_{\mathsf{head}} \top \to \Diamond_{\mathsf{next}} \Diamond_{\mathsf{right}} \top, \tag{57}$$

$$\Diamond_{\mathsf{right}} \top \to \Diamond_{\mathsf{next}} \Diamond_{\mathsf{right}} \top. \tag{58}$$

Then we say that the state of each configuration is encoded in a uniform way over the tape: for all $q \in Q$,

$$\Diamond_q \top \to \Diamond_{\mathsf{next}} \Diamond_q \top \quad \text{and} \quad \Diamond_{\mathsf{next}} \Diamond_q \top \to \Diamond_q \top. \tag{59}$$

Exploiting that q_0 is not reachable from any state, we can say that the tape is initially blank with

$$\Diamond_{q_0} \top \to \Diamond_b \top. \tag{60}$$

Exploiting the commutativity and functionality of next and step, for each transition $\delta(q, a) = (q', a', \mathsf{L})$, we set

$$\Diamond_{\mathsf{next}} (\Diamond_q \top \wedge \Diamond_{\mathsf{head}} \top \wedge \Diamond_a \top) \to \Diamond_{\mathsf{step}} (\Diamond_{q'} \top \wedge \Diamond_{\mathsf{head}} \top \wedge \Diamond_{\mathsf{next}} \Diamond_{a'} \top), \tag{61}$$

and for each transition $\delta(q, a) = (q', a', \mathsf{R})$, we set

$$\Diamond_q \top \wedge \Diamond_{\mathsf{head}} \top \wedge \Diamond_a \top \to \Diamond_{\mathsf{step}} (\Diamond_{a'} \top \wedge \Diamond_{q'} \top \wedge \Diamond_{\mathsf{next}} \Diamond_{\mathsf{head}} \top). \tag{62}$$

We also say that symbols not under the head do not change: for all $a \in \Gamma$, put

$$\Diamond_a \top \wedge \Diamond_{\mathsf{left}} \top \to \Diamond_{\mathsf{step}} \Diamond_a \top, \tag{63}$$

$$\Diamond_a \top \wedge \Diamond_{\mathsf{right}} \top \to \Diamond_{\mathsf{step}} \Diamond_a \top. \tag{64}$$

Let Σ_M^0 be Ξ together with the sp-implications (55)–(64). Finally, we obtain Σ_M from Σ_M^0 by adding the following sp-implication that triggers incompleteness whenever $\Diamond_{q_0} \top \wedge \Diamond_{\mathsf{head}} \top$ is satisfiable in a frame for Σ_M^0:

$$\boldsymbol{\iota}_M = (\Diamond_{q_0} \top \wedge \Diamond_{\mathsf{head}} \top \wedge \Diamond_R\, p \to p), \tag{65}$$

where R is a fresh relation.

CLAIM 51.1. *If M halts on the empty tape, then $\mathsf{SPi} + \Sigma_M$ is complex.*

PROOF. Suppose M halts on the empty tape in $H < \omega$ steps. As $\mathsf{SPi} + \Sigma_M^0$ is complex by Theorem 37 and Proposition 5, it is enough to show that $\mathsf{SPi} + \Sigma_M = \mathsf{SPi} + \Sigma_M^0$. We prove that

$$\{w \mid \mathfrak{M}, w \models \Diamond_{q_0}\top \wedge \Diamond_{\mathsf{head}}\top\} = \emptyset \tag{66}$$

for any model $\mathfrak{M}$ over any frame $\mathfrak{F}$ for Σ_M^0.

Then $\Sigma_M^0 \models_{\mathsf{Kr}} \boldsymbol{\iota}_M$ would follow, and so $\boldsymbol{\iota}_M \in \mathsf{SPi} + \Sigma_M^0$ would hold by the completeness of $\mathsf{SPi} + \Sigma_M^0$.

To prove (66), take any frame $\mathfrak{F} \models \Sigma_M^0$ and suppose to the contrary that there is some $d_{0,0}$ with $\mathfrak{M}, d_{0,0} \models \Diamond_{q_0}\top \wedge \Diamond_{\mathsf{head}}\top$. We show by induction on m that, for any $m \leq H$ and $n < \omega$, there exists $d_{n,m}$ in $\mathfrak{F}$ representing the nth cell in the mth configuration of the computation of M in the following sense: for all $q \in Q$ and $a \in \Gamma$,

(i) $(d_{n,m}, d_{n+1,m}) \in \mathsf{next}$;
(ii) $(d_{n,m}, d_{n,m+1}) \in \mathsf{step}$ whenever $m < H$;
(iii) if the state in the mth configuration is q, then $\mathfrak{M}, d_{n,m} \models \Diamond_q\top$;
(iv) if the nth cell contains a in the mth configuration, then $\mathfrak{M}, d_{n,m} \models \Diamond_a\top$;
(v) if the head is at the nth cell in the mth configuration, then $\mathfrak{M}, d_{n,m} \models \Diamond_{\mathsf{head}}\top$.

Indeed, for $m = n = 0$, (iii)–(v) follow from our assumption and (60). We have $d_{n,0}$ for all $n > 0$ satisfying (i), (iii) and (iv) by (59) and (60). Now suppose inductively that we have $d_{n,m}$ for some $m < H$ and all $n < \omega$. Suppose that in the mth configuration the head is at the nth cell containing symbol a, M is in state q and $\delta(q, a) = (q', a', R)$. (The case when $\delta(q, a) = (q', a', L)$ is similar and left to the reader.) Then, by IH, $\mathfrak{M}, d_{n,m} \models \Diamond_q\top \wedge \Diamond_{\mathsf{head}}\top \wedge \Diamond_a\top$, and so, by (62), there exist $d_{n,m+1}$ and $d_{n+1,m+1}$ such that $(d_{n,m}, d_{n,m+1}) \in \mathsf{step}$, $(d_{n,m+1}, d_{n+1,m+1}) \in \mathsf{next}$, $\mathfrak{M}, d_{n,m+1} \models \Diamond_{a'}\top \wedge \Diamond_{q'}\top$ and $\mathfrak{M}, d_{n+1,m+1} \models \Diamond_{\mathsf{head}}\top$. If $n > 0$ then we have $d_{i,m+1}$ for all $i < n$ satisfying (i), (ii) and (iv) by (52), (55), (56), (63) and (51). We have $d_{i,m+1}$ for all $i > n+1$ satisfying (i) and (ii) by (59), (50) and (52). Then $d_{i,m+1}$ for all $i \geq n+1$ satisfy (iv) by (57), (58), (50) and (64). Finally, we have (iii) by (59) and (50).

Thus, $\mathfrak{M}, d_{n,H} \models \Diamond_{q_h}\top$ for some n, and so the relation R_{q_h} in $\mathfrak{F}$ interpreting $\Diamond_{q_h}$ is not empty, contrary to $\mathfrak{F} \models (53)$. This establishes (66). ⊣

CLAIM 51.2. *If M does not halt on the empty tape, then $\mathsf{SPi} + \Sigma_M$ is incomplete.*

PROOF. Consider the sp-implication

$$\boldsymbol{\iota}' = (\Diamond_{q_0}\top \wedge \Diamond_{\mathsf{head}}\top \wedge p \wedge \Diamond_R\top \to \Diamond_R\, p).$$

On the one hand, it is easy to see that $\{\boldsymbol{\iota}_M\} \models_{\mathsf{Kr}} \boldsymbol{\iota}'$ (cf. Example 1), and so $\Sigma_M \models_{\mathsf{Kr}} \boldsymbol{\iota}'$. On the other hand, take the infinite computation of M starting from the empty tape. Using this computation, we define a frame $\mathfrak{F}$ with domain $W = \{d_{n,m} \mid n, m < \omega\} \cup \{g, g'\}$ by taking:

– $(d_{n,m}, d_{n+1,m}) \in \mathsf{next}$ for all $n, m < \omega$;
– $(d_{n,m}, d_{n,m+1}) \in \mathsf{step}$ for all $n, m < \omega$;
– $(d_{n,m}, g) \in R_q$ if the state of the mth configuration is q, for $q \in Q$;

– $(d_{n,m}, g) \in R_a$ if the nth cell contains a in the mth configuration, for $a \in \Gamma$;
– $(d_{n,m}, g) \in \mathsf{head}$ if the head is at the nth cell in the mth configuration;
– $(d_{n,m}, g) \in \mathsf{left}$ if the head is to the right of the nth cell in the mth configuration;
– $(d_{n,m}, g) \in \mathsf{right}$ if the head is to the left of the nth cell in the mth configuration;
– $(d_{0,0}, g') \in R$.

It is straightforward to check that $\mathfrak{F} \models \Sigma^0_M$. Define an sp-type subalgebra $\mathfrak{A}$ of $\mathfrak{F}^\star$ by taking all subsets of W except those that contain g' but not $d_{0,0}$. Then $\mathfrak{A} \models \Sigma^0_M$. It is easy to see that $\mathfrak{A}$ is a SLO and $\mathfrak{A} \models \boldsymbol{\iota}_M$, and so $\mathfrak{A} \models \Sigma_M$. However, $\mathfrak{A} \not\models \boldsymbol{\iota}'$, witnessed by evaluating p to $\{d_{0,0}\}$. Thus, $\Sigma_M \not\models_{\mathsf{SLO}} \boldsymbol{\iota}'$, and so $\mathsf{SPi} + \Sigma_M$ is incomplete. ⊣

Now, Theorem 51 follows from Claims 51.1 and 51.2. ⊣

QUESTION 6. *Does Theorem 51 hold in the unimodal case? Does it hold for spi-logics with Horn correspondents?*

§9. Some related topics.

9.1. Spi-definability. A class $\mathcal{C}$ of frames is called *spi-definable* if $\mathcal{C} = \mathsf{Kr}_\Sigma$, for some set Σ of sp-implications. In this section, we prove a necessary condition for spi-definability and use it to give a few examples of modally definable frame classes that are not spi-definable. To keep the notation simple, we formulate everything for the unimodal setting only, that is, for $\mathcal{R} = \{R\}$.

Suppose that $\mathfrak{F}_i = (W_i, R_i)$, for $i \in I$, $\mathfrak{G} = (W, R^{\mathfrak{G}})$, $\mathfrak{T} = (T, R^{\mathfrak{T}})$ are frames, $w \in T$, $g_i \colon \mathfrak{T} \to \mathfrak{F}_i$, for $i \in I$, and $h \colon \mathfrak{T} \to \mathfrak{G}$ are homomorphisms, and that $Z \subseteq \big(\prod_{i\in I} W_i\big) \times W$. We write

$$(\mathfrak{F}_i, g_i)_{i\in I} >>_Z (\mathfrak{G}, h, w)$$

if the following conditions hold:

(s1) $\big((g_i(w))_{i\in I}, h(w)\big) \in Z$;
(s2) for all $(\boldsymbol{x}, y) \in Z$ and $\boldsymbol{x}' = (x_i' \in W_i \mid i \in I)$, if $(x_i, x_i') \in R_i$ for all $i \in I$, then there is y' such that $(y, y') \in R^{\mathfrak{G}}$ and $(\boldsymbol{x}', y') \in Z$;
(s3) for all $(\boldsymbol{x}, y) \in Z$ and $A \subseteq T$, if $x_i \in g_i[A]$ for all $i \in I$, then $y \in h[A]$.

We write

$$(\mathfrak{F}_i)_{i\in I} >> \mathfrak{G}$$

if, for all finite trees $\mathfrak{T}$ with root w and all homomorphisms $h \colon \mathfrak{T} \to \mathfrak{G}$, there exist $(g_i)_{i\in I}$ and Z such that $(\mathfrak{F}_i, g_i)_{i\in I} >>_Z (\mathfrak{G}, h, w)$.

THEOREM 52. *For any sp-implication $\boldsymbol{\iota}$, if $(\mathfrak{F}_i)_{i\in I} >> \mathfrak{G}$ and $\mathfrak{F}_i \models \boldsymbol{\iota}$ for all $i \in I$, then $\mathfrak{G} \models \boldsymbol{\iota}$.*

PROOF. Suppose $\boldsymbol{\iota} = (\sigma \to \tau)$. It is enough to show that, for the correspondent $\Psi_{\boldsymbol{\iota}}$ of $\boldsymbol{\iota}$ from (18)–(19), $\mathfrak{G} \models \Psi_{\boldsymbol{\iota}}$ holds whenever $\mathfrak{F}_i \models \Psi_{\boldsymbol{\iota}}$ for $i \in I$. Recall the respective tree models $\mathfrak{M}_\sigma$ and $\mathfrak{M}_\tau$ from §4.2.1, and let $W_\sigma = \{v_0, \dots, v_{n_\sigma}\}$ with $v_0 = r_\sigma$. Let $x_0, \dots, x_{n_\sigma}$ be a sequence of points in $\mathfrak{G}$ such that $(x_k, x_\ell) \in R^{\mathfrak{G}}$

whenever $(v_k, v_\ell) \in R_\sigma$. Then $h^\sigma\colon \mathfrak{T}_\sigma \to \mathfrak{G}$ defined by $h^\sigma(v_k) = x_k$, for $k \le n_\sigma$, is a homomorphism. As $(\mathfrak{F}_i)_{i\in I} >> \mathfrak{G}$, there are $(g_i^\sigma)_{i\in I}$ and Z such that

$$g_i^\sigma\colon \mathfrak{T}_\sigma \to \mathfrak{F}_i \text{ are homomorphisms for all } i \in I, \tag{67}$$

$$(\mathfrak{F}_i, g_i^\sigma)_{i\in I} >>_Z (\mathfrak{G}, h^\sigma, r_\sigma). \tag{68}$$

Since $\mathfrak{F}_i \models \Psi_{\boldsymbol{\iota}}$, it follows that $\mathfrak{F}_i \models \Psi'_{\boldsymbol{\iota}}[g_i^\sigma(r_\sigma)/\hat{v}_0]$ for all $i \in I$, and so, by (67), there exist homomorphisms $g_i^\tau\colon \mathfrak{T}_\tau \to \mathfrak{F}_i$ such that

$$g_i^\tau(r_\tau) = g_i^\sigma(r_\sigma) \wedge \bigwedge_{\substack{(u,p)\\ u\in\mathfrak{v}_\tau(p)}} \bigvee_{v\in\mathfrak{v}_\sigma(p)} \big(g_i^\tau(u) = g_i^\sigma(v)\big), \quad \text{for all } i \in I. \tag{69}$$

We define a homomorphism $h^\tau\colon \mathfrak{T}_\tau \to \mathfrak{G}$ such that

$$\big((g_i^\tau(u))_{i\in I}, h^\tau(u)\big) \in Z, \quad \text{for all } u \in W_\tau \tag{70}$$

in a step-by-step manner, by constructing its approximations $f_0, f_1, f_2, \dots$ with domains $B_0, B_1, \dots$ which are subsets of W_τ and initial segments of $\mathfrak{T}_\tau$. To begin with, let $B_0 = \{r_\tau\}$ and $f_0 = \big\{\big(r_\tau, h^\sigma(r_\sigma)\big)\big\}$. By (69) and **(s1)** of (68), $\big((g_i^\tau(r_\tau))_{i\in I}, f_0(r_\tau)\big) = \big((g_i^\sigma(r_\sigma))_{i\in I}, h^\sigma(r_\sigma)\big) \in Z$. So suppose B_l and f_l are defined for some l, and we have $\big((g_i^\tau(u))_{i\in I}, f_l(u)\big) \in Z$ for all $u \in B_l$ (IH). Take some $x \in B_l$ and $y \notin B_l$ such that $(x, y) \in R_\tau$. Since all g_i^τ are homomorphisms, we have $\big(g_i^\tau(x), g_i^\tau(y)\big) \in R_i$. By IH, $\big((g_i^\tau(x))_{i\in I}, f_l(x)\big) \in Z$, and so, by **(s2)** of (68), there is $z \in W$ such that $\big(f_l(x), z\big) \in R^{\mathfrak{G}}$ and $\big((g_i^\tau(y))_{i\in I}, z\big) \in Z$. Thus, we may extend B_l and f_l by setting $B_{l+1} = B_l \cup \{y\}$ and $f_{l+1} = f_l \cup \{(y, z)\}$ while preserving IH. Clearly, $h^\tau = \bigcup_{l<\omega} f_l$ is a homomorphism as required in (70).

Suppose $W_\tau = \{u_0, \dots, u_{n_\tau}\}$ with $u_0 = r_\tau$. We claim that

$$\mathfrak{G} \models \Big((\hat{v}_0 = \hat{u}_0) \wedge \bigwedge_{\substack{k,\ell\le n_\tau,\\ (u_k,u_\ell)\in R_\tau}} R(\hat{u}_k, \hat{u}_\ell) \wedge \bigwedge_{\substack{k\le n_\tau,\, p\in\mathsf{var},\\ u_k\in\mathfrak{v}_\tau(p)}} \bigvee_{\substack{\ell\le n_\tau,\\ v_\ell\in\mathfrak{v}_\sigma(p)}} (\hat{u}_k = \hat{v}_\ell)\Big)\big[h^\sigma(\boldsymbol{v})/\hat{\boldsymbol{v}}, h^\tau(\boldsymbol{u})/\hat{\boldsymbol{u}}\big], \tag{71}$$

proving $\mathfrak{G} \models \Psi_{\boldsymbol{\iota}}$. Indeed, $h^\sigma(r_\sigma) = h^\tau(r_\tau)$ and h^τ is a homomorphism, so it is enough to show the second line in (71). Fix u_k and p such that $u_k \in \mathfrak{v}_\tau(p)$. By (69), for any $i \in I$, there is $v_i \in \mathfrak{v}_\sigma(p)$ with $g_i^\tau(u_k) = g_i^\sigma(v_i)$, and so $g_i^\tau(u_k) \in g_i^\sigma[\mathfrak{v}_\sigma(p)]$ for all $i \in I$. By (70) and **(s3)** of (68), we have $h^\tau(u_k) \in h^\sigma[\mathfrak{v}_\sigma(p)]$, and so there is some $v_\ell \in \mathfrak{v}_\sigma(p)$ with $h^\tau(u_k) = h^\sigma(v_\ell)$, as required in (71). ⊣

In certain cases, we may simplify the criterion of the previous theorem:

Proposition 53. *If there exist homomorphisms $f_i\colon \mathfrak{G} \to \mathfrak{F}_i$, for $i \in I$, and Z such that $(\mathfrak{F}_i, f_i)_{i\in I} >>_Z (\mathfrak{G}, id, v)$, for all v in $\mathfrak{G}$ and the identity map $id\colon \mathfrak{G} \to \mathfrak{G}$, then $(\mathfrak{F}_i)_{i\in I} >> \mathfrak{G}$.*

Proof. Suppose $h\colon \mathfrak{T} \to \mathfrak{G}$ is a homomorphism, for a finite tree $\mathfrak{T}$ with root w. Let $v = h(w)$. By our assumption, there are homomorphisms $f_i\colon \mathfrak{G} \to \mathfrak{F}_i$, for $i \in I$, and Z such that $(\mathfrak{F}_i, f_i)_{i\in I} >>_Z (\mathfrak{G}, id, v)$. Define $g_i\colon \mathfrak{T} \to \mathfrak{F}_i$ by $g_i = f_i \circ h$, for $i \in I$. Then it is not hard to check that $(\mathfrak{F}_i, g_i)_{i\in I} >>_Z (\mathfrak{G}, h, w)$. ⊣

A relation R is called *pseudo-transitive* if

$$\forall x, y, z\, \big(R(x, y) \wedge R(y, z) \to R(x, z) \vee (x = z)\big);$$

R is *pseudo-equivalence* if it is symmetric and pseudo-transitive. Pseudo-equivalence relations are the frames for the modal logic Diff, also characterised by the $\neq$ relation on nonempty sets.

PROPOSITION 54. *Neither the class of all pseudo-transitive nor the class of all pseudo-equivalence frames is spi-definable.*

PROOF. Take the frames $\mathfrak{F}_1$, $\mathfrak{F}_2$ and $\mathfrak{G}$ in Fig. 13. We show that the conditions of Proposition 53 hold, and so $(\mathfrak{F}_1, \mathfrak{F}_2) >\!\!> \mathfrak{G}$. Consider the homomorphism

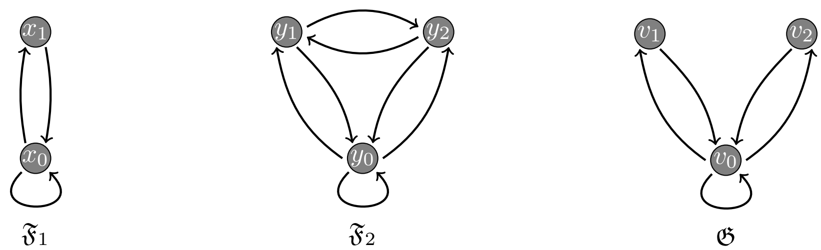


FIGURE 13. Frames showing spi-undefinability of pseudo-transitivity.

$f_1 : \mathfrak{G} \to \mathfrak{F}_1$ where $f_1(v_0) = x_0$, $f_1(v_1) = f_1(v_2) = x_1$, and the homomorphism $f_2 : \mathfrak{G} \to \mathfrak{F}_2$ where $f_2(v_i) = y_i$ for $i \leq 2$, and let

$$Z = \{(x_0, y_0, v_0), (x_1, y_1, v_1), (x_1, y_2, v_2), (x_0, y_1, v_0), (x_0, y_2, v_0), (x_1, y_0, v_0)\}.$$

We claim that, for all $i \leq 2$, we have $\big((\mathfrak{F}_1, f_1), (\mathfrak{F}_2, f_2)\big) >\!\!>_Z (\mathfrak{G}, id, v_i)$. Indeed, **(s1)** clearly holds. It is easy to check that **(s2)** holds, because

- for all $(x, y) \in \mathfrak{F}_1 \times \mathfrak{F}_2$, there is $v \in \mathfrak{G}$ with $(x, y, v) \in Z$,
- for all v in $\mathfrak{G}$, we have $(v, v_0) \in R^{\mathfrak{G}}$ and $(v_0, v) \in R^{\mathfrak{G}}$.

Finally, we leave it to the reader to consider all 7 possible cases for the non-empty set $A \subseteq \{v_0, v_1, v_2\}$ and show **(s3)**. $\dashv$

Recall that a relation R is called *weakly connected* if

$$\forall x, y, z \,\big(R(x, y) \wedge R(x, z) \to R(y, z) \vee R(z, y) \vee (y = z)\big).$$

Transitive and weakly connected relations are the frames for the modal logic $\mathsf{K4.3}$. Note that the class of reflexive, transitive and weakly connected relations—linear quasiorders, the frames for $\mathsf{S4.3}$—is spi-definable; see Σ_{lin} in (46).

PROPOSITION 55. *Neither the class of all weakly connected nor the class of all transitive and weakly connected frames is spi-definable.*

PROOF. Take the frames $\mathfrak{F}_1$, $\mathfrak{F}_2$ and $\mathfrak{G}$ in Fig. 14. We show that the conditions of Proposition 53 hold, and so $(\mathfrak{F}_1, \mathfrak{F}_2) >\!\!> \mathfrak{G}$. Consider the homomorphism $f_1 \colon \mathfrak{G} \to \mathfrak{F}_1$, where $f_1(v_0) = x_0$, $f_1(v_1) = f_1(v_2) = x_1$, and the homomorphism $f_2 \colon \mathfrak{G} \to \mathfrak{F}_2$, where $f_2(v_i) = y_i$ for $i \leq 2$, and let

$$Z = \{(x_0, y_0, v_0), (x_1, y_1, v_1), (x_1, y_2, v_2)\}.$$

Then it is easy to check that $\big((\mathfrak{F}_1, f_1), (\mathfrak{F}_2, f_2)\big) >\!\!>_Z (\mathfrak{G}, id, v_i)$, for $i \leq 2$. $\dashv$

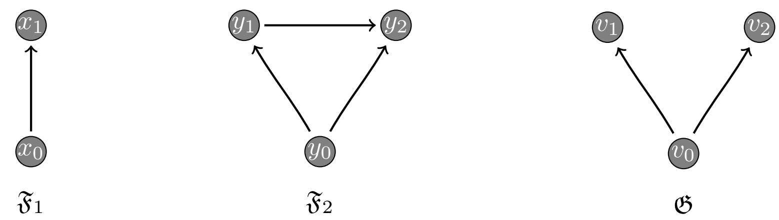


FIGURE 14. Frames showing spi-undefinability of weak connectedness.

A relation R is called *confluent* if

$$\forall x, y, z \, \big(R(x, y) \land R(x, z) \to \exists u \, \big(R(y, u) \land R(z, u)\big)\big).$$

Transitive and confluent relations are the frames for the modal logic $\mathsf{K4.2}$.

PROPOSITION 56. *Neither the class of all confluent nor the class of all transitive and confluent frames is spi-definable.*

PROOF. Take the frames $\mathfrak{F}$ and $\mathfrak{G}$ in Fig. 15. We show that the conditions of Proposition 53 hold, and so $(\mathfrak{F}) >> \mathfrak{G}$. Consider the homomorphism $f : \mathfrak{G} \to \mathfrak{F}$,

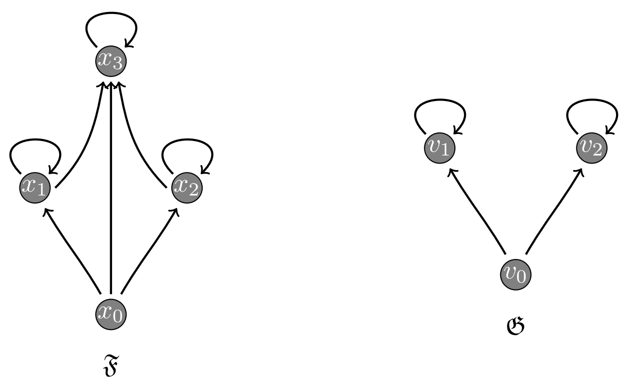


FIGURE 15. Frames showing spi-undefinability of confluence.

where $f(v_i) = x_i$ for $i \leq 2$, and let

$$Z = \{(x_0, v_0), (x_1, v_1), (x_2, v_2), (x_3, v_1), (x_3, v_2)\}.$$

Then it is easy to check that $(\mathfrak{F}, f) >>_Z (\mathfrak{G}, id, v_i)$, for $i \leq 2$. $\dashv$

We say that a relation R *has the McKinsey property* if

$$\forall x \exists y \, \big(R(x, y) \land \forall z \, \big(R(y, z) \to (y = z)\big)\big).$$

Transitive relations with this property are the frames for the modal logic $\mathsf{K4.1}$.

PROPOSITION 57. *The class of all transitive frames with the McKinsey property is not spi-definable.*

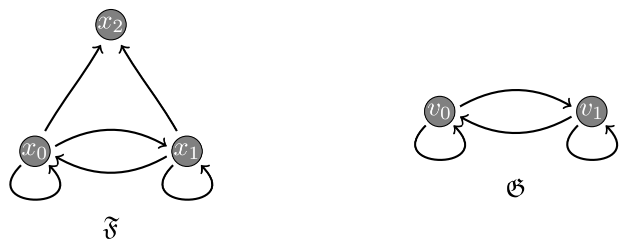


FIGURE 16. Frames showing spi-undefinability of transitive frames with the McKinsey property.

PROOF. Take the frames $\mathfrak{F}$ and $\mathfrak{G}$ in Fig. 16. We show that the conditions of Proposition 53 hold, and so $(\mathfrak{F}) >> \mathfrak{G}$. Consider the homomorphism $f\colon \mathfrak{G} \to \mathfrak{F}$, where $f(v_i) = x_i$ for $i \leq 1$, and let

$$Z = \{(x_0, v_0), (x_1, v_1), (x_2, v_0), (x_2, v_1)\}.$$

Then it is easy to check that $(\mathfrak{F}, f) >>_Z (\mathfrak{G}, id, v_i)$, for $i \leq 1$. ⊣

As mentioned above, the class of linear quasiorders is spi-definable. However, confluent quasiorders (the frames for the modal logic $\mathsf{S4.2}$) and quasiorders with the McKinsey property (the frames for the modal logic $\mathsf{S4.1}$) are not spi-definable, which is a consequence of the following:

PROPOSITION 58. *Every unimodal spi-logic $L \supseteq \mathsf{SPi}_{qo}$ is a subframe spi-logic.*

PROOF. We show that, for every sp-implication $\boldsymbol{\iota} = (\sigma \to \tau)$, if $\mathfrak{F} \not\models \boldsymbol{\iota}$ and $\mathfrak{F} = (W, R)$ is a subframe of some quasiorder $\mathfrak{F}' = (W', R')$, then $\mathfrak{F}' \not\models \boldsymbol{\iota}$. Let $\mathfrak{M} = (\mathfrak{F}, \mathfrak{v})$ be such that $\mathfrak{M}, w \not\models \boldsymbol{\iota}$, for some $w \in W$, and let $\mathfrak{M}' = (\mathfrak{F}', \mathfrak{v})$. By induction on the construction of an sp-formula ϱ, we show that $\{u \mid \mathfrak{M}, u \models \varrho\} = \{u \mid \mathfrak{M}', u \models \varrho\} \cap W$, and so $\mathfrak{M}', w \not\models \boldsymbol{\iota}$. The basis of induction follows from the definition, and the cases of $\top$ and $\wedge$ are trivial. Let $\varrho = \Diamond \varrho'$. By IH, $\{u \mid \mathfrak{M}, u \models \varrho\} \subseteq \{u \mid \mathfrak{M}', u \models \varrho\} \cap W$. For the converse inclusion, there are four cases. The case $\varrho' = \top$ is trivial as R is reflexive. Now, let $u \in W$ be such that $\mathfrak{M}', u \models \varrho$. Then $\mathfrak{M}', v \models \varrho'$, for some $v \in W'$ with $(u, v) \in R'$. If ϱ' is a variable, then $v \in W$ and $\mathfrak{M}, v \models \varrho'$ by the definition of $\mathfrak{M}'$, and so $\mathfrak{M}', u \models \varrho$. If $\varrho' = \Diamond \pi$ then, by transitivity of R', $\mathfrak{M}', u \models \Diamond \pi$, and so, by IH, $\mathfrak{M}, u \models \varrho'$, from which, in view of reflexivity of R, we obtain $\mathfrak{M}, u \models \varrho$. Finally, let $\varrho' = \pi_1 \wedge \cdots \wedge \pi_n$, where none of the π_i is a conjunction or $\top$. If one of them is a variable, then $v \in W$ and we are done by IH. And if $\pi_i = \Diamond \pi_i'$ for all i then, by transitivity of R', $\mathfrak{M}', u \models \Diamond \pi_i'$ for all i, and we obtain $\mathfrak{M}, u \models \varrho$ by IH and reflexivity of R. ⊣

9.2. Spi-logics with $\perp$. One can introduce a limited form of negation to the language of sp-formulas by adding the 'falsehood' constant $\perp$ (such that $\mathfrak{M}, w \not\models \perp$ for any point w in any Kripke model $\mathfrak{M}$). We call the sp-formulas of this extended language *sp$^\perp$-formulas*, and define *sp$^\perp$-implications* accordingly. A class $\mathcal{C}$ of frames is *spi$^\perp$-definable* if $\mathcal{C} = \mathsf{Kr}_\Sigma$, for some set Σ of sp$^\perp$-implications.

PROPOSITION 59. *A class of frames is spi-definable iff it is spi$^\perp$-definable.*

PROOF. Suppose $\mathcal{C} = \mathsf{Kr}_\Sigma$, for some set Σ of sp$^\perp$-implications. As sp$^\perp$-implications $\sigma \to \tau$ hold in all frames whenever σ contains $\perp$, we may assume that $\bot$ only occurs in τ. Then it is easy to see that, for every frame $\mathfrak{F}$, we have $\mathfrak{F} \models \sigma \to \tau$ iff $\mathfrak{F} \models \sigma \to p$, where p is a fresh variable not occurring in σ. $\dashv$

All the notions introduced above can be extended to $\mathrm{sp}^{\bot}$. Thus, a structure $\mathfrak{A} = (A, \wedge, \bot, \top, \Diamond_R)_{R\in\mathcal{R}}$ is called an *$\mathrm{sp}^{\bot}$-type algebra* (*of signature* $\mathcal{R}$). Given $\mathrm{sp}^{\bot}$-type algebras $\mathfrak{A}$ and $\mathfrak{B}$ of the same signature, a function $\eta\colon \mathfrak{A} \to \mathfrak{B}$ is an $\mathrm{sp}^{\bot}$-*embedding* if it is an sp-embedding and $\eta(\bot) = \bot$. We call $\mathfrak{A}$ a *bounded meet-semilattice with normal monotone operators* (or $\mathrm{SLO}^{\bot}$) if $(A, \wedge, \top, \Diamond_R)_{R\in\mathcal{R}}$ is a SLO with $\leq$-smallest element $\bot$, and $\Diamond_R \bot = \bot$ for $R \in \mathcal{R}$. The set of $\mathrm{sp}^{\bot}$-implications that are valid in all $\mathrm{SLO}^{\bot}$s is denoted by $\mathsf{SPi}^{\bot}$. For a set Σ of $\mathrm{sp}^{\bot}$-implications, $\mathsf{SLO}^{\bot}_{\Sigma}$ denotes the class of $\mathrm{SLO}^{\bot}$s validating Σ. We set

$$\Sigma \models_{\mathsf{SLO}^{\bot}} \boldsymbol{\iota} \quad \text{iff} \quad \mathfrak{A} \models \boldsymbol{\iota} \ \text{ for every } \mathfrak{A} \in \mathsf{SLO}^{\bot}_{\Sigma}.$$

(Note that $\Sigma \models_{\mathsf{SLO}^{\bot}}$ can be captured syntactically by adding the axioms $\bot \to p$ and $\Diamond_R \bot \to \bot$, for $R \in \mathcal{R}$, to the calculus in (8)–(9).) For any set Σ of $\mathrm{sp}^{\bot}$-implications, we define the *$\mathrm{spi}^{\bot}$-logic* $\mathsf{SPi}^{\bot} + \Sigma$ *axiomatised by* Σ as

$$\mathsf{SPi}^{\bot} + \Sigma = \{\boldsymbol{\iota} \mid \boldsymbol{\iota} \text{ is an } \mathrm{sp}^{\bot}\text{-implication and } \Sigma \models_{\mathsf{SLO}^{\bot}} \boldsymbol{\iota}\}.$$

Now one can define the notions of completeness, complexity, finite frame property in the same way as in the sp-case. We give examples of incomplete spi-logics $\mathsf{SPi} + \Sigma$ such that $\mathsf{SPi}^{\bot} + \Sigma$ is a complete or even complex $\mathrm{spi}^{\bot}$-logic.

EXAMPLE 60. By Theorem 31, $\mathsf{SPi} + \Sigma$ for $\Sigma = \{p \to \Diamond p, \Diamond p \to \Diamond q\}$ is an incomplete spi-logic. However, only the one-element $\mathrm{SLO}^{\bot}$ can validate the $\mathrm{spi}^{\bot}$-logic $\mathsf{SPi}^{\bot} + \Sigma$, and so $\Sigma \models_{\mathsf{SLO}^{\bot}} \boldsymbol{\iota}$ for every $\mathrm{sp}^{\bot}$-implication $\boldsymbol{\iota}$. Thus, $\mathsf{SPi}^{\bot} + \Sigma$ is a complete $\mathrm{spi}^{\bot}$-logic. By Theorem 34, $\mathsf{SPi} + \{\Diamond_R \Diamond_S p \to q\}$ is an incomplete spi-logic. However, using a proof similar to that of Theorem 33, one can readily show that $\mathsf{SPi}^{\bot} + \{\Diamond_R \Diamond_S p \to q\}$ is a complex $\mathrm{spi}^{\bot}$-logic.

On the other hand, completeness and complexity do transfer from sp to $\mathrm{sp}^{\bot}$:

PROPOSITION 61. *Let Σ be a set of sp-implications.*

(i) *If the spi-logic $\mathsf{SPi}+\Sigma$ is complete, then the $\mathrm{spi}^{\bot}$-logic $\mathsf{SPi}^{\bot}+\Sigma$ is complete.*
(ii) *If the spi-logic $\mathsf{SPi}+\Sigma$ is complex, then the $\mathrm{spi}^{\bot}$-logic $\mathsf{SPi}^{\bot}+\Sigma$ is complex.*

PROOF. (i) Suppose $\Sigma \models_{\mathsf{Kr}} \boldsymbol{\iota}$ for some $\mathrm{sp}^{\bot}$-implication $\boldsymbol{\iota}$ containing $\bot$. Then we may assume that $\boldsymbol{\iota}$ is of the form $\sigma \to \bot$, in which case $\Sigma \models_{\mathsf{Kr}} \sigma \to p$, for a fresh variable p. Also, $\Sigma \models_{\mathsf{Kr}} \Diamond_R \sigma \to p$ for every $\Diamond_R$ occurring in Σ, whence $\Sigma \models_{\mathsf{SLO}} \sigma \to p$ and $\Sigma \models_{\mathsf{SLO}} \Diamond_R \sigma \to p$. So, in every $\mathfrak{A} \in \mathsf{SLO}_{\Sigma}$, there is a $\leq$-smallest element $\bot$, for which $\Diamond_R \bot = \bot$ for every $\Diamond_R$ occurring in Σ. This shows that $\Sigma \models_{\mathsf{SLO}^{\bot}} \sigma \to \bot$.

(ii) Suppose $\mathfrak{A} \in \mathsf{SLO}^{\bot}_{\Sigma}$. Then the sp-type reduct $\mathfrak{A}^{\downarrow}$ of $\mathfrak{A}$ is in SLO_{Σ}, and so there is an sp-embedding $f\colon \mathfrak{A}^{\downarrow} \to \mathfrak{F}^{\star}$ for some $\mathfrak{F} = (W, R^{\mathfrak{F}})_{R\in\mathcal{R}}$ with $\mathfrak{F} \models \Sigma$. Let $V = W \setminus f(\bot)$ and $R^{\mathfrak{F}}_V = R^{\mathfrak{F}} \cap (V \times V)$, for $R \in \mathcal{R}$. Then it is easy to see that the frame $\mathfrak{G} = (V, R^{\mathfrak{F}}_V)_{R\in\mathcal{R}}$ is a generated subframe of $\mathfrak{F}$ (and so $\mathfrak{G} \models \Sigma$), and the map $g\colon \mathfrak{A} \to \mathfrak{G}^{\star\bot}$ defined by $g(a) = f(a) \setminus f(\bot)$ is an $\mathrm{sp}^{\bot}$-embedding. $\dashv$

A complete (complex) $\mathrm{spi}^{\bot}$-logic can always be turned into a complete (complex) spi-logic, using a fresh diamond operator:

THEOREM 62. *Let Σ be a set of spi$^\perp$-implications not using $\Diamond_R$. Let Σ' be obtained from Σ by replacing each occurrence of $\perp$ by $\Diamond_R \top$ and adding $\Diamond_R \top \to p$ and $\Diamond_S \Diamond_R \top \to \Diamond_R \top$, for each $\Diamond_S$ occurring in Σ. Then $\mathsf{SPi}^\perp + \Sigma$ has property P iff $\mathsf{SPi} + \Sigma'$ has property P, where P stands for any of the following: 'is complete', 'is complex', 'has the finite frame property', 'is decidable'.*

PROOF. Let $\mathcal{R}_\Sigma = \{S \mid \Diamond_S \text{ occurs in } \Sigma\}$. Given an sp$^\perp$-implication $\boldsymbol{\iota}$ using only $\Diamond_S$, for $S \in \mathcal{R}_\Sigma$, denote by $\boldsymbol{\iota}^\uparrow$ the sp-implication obtained by replacing each occurrence of $\perp$ in Σ by $\Diamond_R \top$. Similarly, for any $\mathfrak{A} \in \mathsf{SLO}^\perp_\Sigma$, denote by $\mathfrak{A}^\uparrow$ the sp-type reduct of $\mathfrak{A}$ with an additional operator $\Diamond_R$ for which $\Diamond_R a = \perp$ for all $a \in A$. Then $\mathfrak{A}^\uparrow \in \mathsf{SLO}_{\Sigma'}$, and $\mathfrak{A} \models \boldsymbol{\iota}$ iff $\mathfrak{A}^\uparrow \models \boldsymbol{\iota}^\uparrow$. Conversely, given an sp-implication $\boldsymbol{\iota}$ using only $\Diamond_S$, for $S \in \mathcal{R}_\Sigma \cup \{R\}$, denote by $\boldsymbol{\iota}^\downarrow$ the sp$^\perp$-implication obtained by replacing each maximal subformula of the form $\Diamond_R \varrho$ in $\boldsymbol{\iota}$ with $\perp$. Observe that in any $\mathfrak{A} \in \mathsf{SLO}_{\Sigma'}$, $\Diamond_R \top$ is the $\leq$-smallest element with $\Diamond_S \Diamond_R \top = \Diamond_R \top$ for all $S \in \mathcal{R}_\Sigma$. Denote by $\mathfrak{A}^\downarrow$ the result of removing $\Diamond_R$ from $\mathfrak{A}$ and setting $\perp = \Diamond_R \top$. Then $\mathfrak{A}^\downarrow \in \mathsf{SLO}^\perp_\Sigma$, and $\mathfrak{A} \models \boldsymbol{\iota}$ iff $\mathfrak{A}^\downarrow \models \boldsymbol{\iota}^\downarrow$. It remains to observe that, for any frame $\mathfrak{F} = (W, S^\mathfrak{F})_{S \in \mathcal{R}_\Sigma}$, we have $\mathfrak{F} \models \Sigma$ iff $(W, S^\mathfrak{F}, \emptyset)_{S \in \mathcal{R}_\Sigma} \models \Sigma'$, and $R^\mathfrak{F} = \emptyset$ follows whenever $(W, S^\mathfrak{F}, R^\mathfrak{F})_{S \in \mathcal{R}_\Sigma} \models \Sigma'$. With these observations, all the statements of the theorem are straightforward. $\dashv$

9.3. Spi-rules. An *spi-rule*, $\boldsymbol{\rho}$, takes the form $\frac{\boldsymbol{\iota}_1, \dots, \boldsymbol{\iota}_n}{\boldsymbol{\iota}}$, where $\boldsymbol{\iota}_1, \dots, \boldsymbol{\iota}_n, \boldsymbol{\iota}$ are sp-implications. We identify the rule $\frac{\emptyset}{\boldsymbol{\iota}}$ with $\boldsymbol{\iota}$. We say that an spi-rule $\boldsymbol{\rho} = \frac{\boldsymbol{\iota}_1, \dots, \boldsymbol{\iota}_n}{\boldsymbol{\iota}}$ *holds* in a Kripke model $\mathfrak{M}$ and write $\mathfrak{M} \models \boldsymbol{\rho}$ if $\mathfrak{M} \models \boldsymbol{\iota}$ whenever $\mathfrak{M} \models \boldsymbol{\iota}_i$ for $1 \leq i \leq n$. We say that $\boldsymbol{\rho}$ is *valid* in a frame $\mathfrak{F}$ and write $\mathfrak{F} \models \boldsymbol{\rho}$ if $\boldsymbol{\rho}$ holds in every Kripke model based on $\mathfrak{F}$. Given a set Θ of spi-rules, we write $\mathfrak{F} \models \Theta$ whenever $\mathfrak{F} \models \boldsymbol{\rho}$ for every $\boldsymbol{\rho} \in \Theta$ and set $\mathsf{Kr}_\Theta = \{\mathfrak{F} \mid \mathfrak{F} \models \Theta\}$.

We say that $\boldsymbol{\rho}$ is *valid* in an algebra $\mathfrak{A}$ having an sp-type reduct and write $\mathfrak{A} \models \boldsymbol{\rho}$ if $\mathfrak{A}$ validates the sp-type *quasiequation*

$$(\boldsymbol{\iota}_1^* \,\&\, \dots \,\&\, \boldsymbol{\iota}_n^*) \Rightarrow \boldsymbol{\iota}^*,$$

where $(\sigma \to \tau)^* = (\sigma \wedge \tau \approx \sigma)$: for any valuation $\mathfrak{a}$ in $\mathfrak{A}$, whenever $\mathfrak{A} \models \boldsymbol{\iota}_i[\mathfrak{a}]$ for all i $(1 \leq i \leq n)$, then $\mathfrak{A} \models \boldsymbol{\iota}[\mathfrak{a}]$. A set *rL* of spi-rules is called an *spi-rule logic* if $rL = \{\boldsymbol{\rho} \mid \mathfrak{A} \models \boldsymbol{\rho}$ for every $\mathfrak{A} \in \mathcal{C}\}$ for some class $\mathcal{C}$ of SLOs. Given an spi-rule logic *rL*, we write $\mathfrak{A} \models rL$ if $\mathfrak{A} \models \boldsymbol{\rho}$ for any $\boldsymbol{\rho} \in rL$. For a class $\mathcal{C}$ of algebras with sp-type reducts, let $\mathcal{C}_{rL} = \{\mathfrak{A} \in \mathcal{C} \mid \mathfrak{A} \models rL\}$. We say that an spi-rule $\boldsymbol{\rho}$ *follows from rL over* $\mathcal{C}$ and write $rL \models_\mathcal{C} \boldsymbol{\rho}$ if $\mathfrak{A} \models \boldsymbol{\rho}$, for any $\mathfrak{A} \in \mathcal{C}_{rL}$. We call *rL*

- $\mathcal{C}$-*embeddable* if every $\mathfrak{A} \in \mathsf{SLO}_{rL}$ is embeddable into the sp-type reduct of some $\mathfrak{B} \in \mathcal{C}_{rL}$;
- $\mathcal{C}$-*rule-conservative* if $rL \models_\mathcal{C} \boldsymbol{\rho}$ implies $rL \models_{\mathsf{SLO}} \boldsymbol{\rho}$, for every spi-rule $\boldsymbol{\rho}$;
- $\mathcal{C}$-*conservative* if $rL \models_\mathcal{C} \boldsymbol{\iota}$ implies $rL \models_{\mathsf{SLO}} \boldsymbol{\iota}$, for every sp-implication $\boldsymbol{\iota}$.

In particular, let

$$\mathsf{CA} = \{\mathfrak{F}^\star \mid \mathfrak{F} \text{ is a frame}\}, \qquad \mathsf{BAO} = \{\mathfrak{A} \mid \mathfrak{A} \text{ is a BAO}\}.$$

Extending the corresponding notions for spi-logics, we call an spi-rule logic *rL*

- *complex* if it is CA-embeddable;
- *globally complete* if it is CA-rule-conservative;
- *complete* if it is CA-conservative.

As quasiequations are preserved under taking subalgebras, we always have:

$$\mathcal{C}\text{-embeddable} \Rightarrow \mathcal{C}\text{-rule-conservative} \Rightarrow \mathcal{C}\text{-conservative}. \tag{72}$$

Also, since $\mathfrak{F}^\star$ is the sp-type reduct of some BAO, we have:

$$\begin{array}{ccccc} \text{complex} & \Rightarrow & \text{globally complete} & \Rightarrow & \text{complete} \\ \Downarrow & & \Downarrow & & \Downarrow \\ \mathsf{BAO}\text{-embeddable} & \Rightarrow & \mathsf{BAO}\text{-rule-conservative} & \Rightarrow & \mathsf{BAO}\text{-conservative.} \end{array}$$

LEMMA 63. *For any spi-rule logic rL, if rL is* BAO*-rule-conservative, then rL is* BAO*-embeddable.*

PROOF. Suppose rL is BAO-rule-conservative and $\mathfrak{A} \in \mathsf{SLO}_{rL}$. To embed $\mathfrak{A}$ into the sp-type reduct of some $\mathfrak{B} \in \mathsf{BAO}_{rL}$, take the diagram $D_\mathfrak{A}$ of $\mathfrak{A}$, that is, the set all literals—equations and negated equations—that hold in $\mathfrak{A}$ and are built from the elements of $\mathfrak{A}$ as constants using the sp-type operations. For any finite set X of literals of this extended type, we write $X(a_1, \dots, a_n)$ to indicate that the $\mathfrak{A}$-type constants occurring in the literals in X are among $a_1, \dots, a_n$. If $X = \{\varphi\}$, we write $\varphi(a_1, \dots, a_n)$. We write $\varphi(p_1/a_1, \dots, p_n/a_n)$ for the spi-type literal where the constants a_i in φ are simultaneously replaced by variables p_i.

CLAIM 63.1. *For any finite subset $X(a_1, \dots, a_n)$ of $D_\mathfrak{A}$, there exist $\mathfrak{B}^X \in \mathsf{BAO}$ and elements $a_1^X, \dots, a_n^X$ in $\mathfrak{B}^X$ such that $\mathfrak{B}^X \models rL$ and*

$$\mathfrak{B}^X \models \bigwedge_{\varphi \in X} \varphi(p_1/a_1, \dots, p_n/a_n)[a_1^X, \dots, a_n^X]. \tag{73}$$

PROOF. If all literals in X are equations, then we can take $\mathfrak{B}^X$ to be the one-element BAO (for which $\mathfrak{B}^X \models rL$ for any rL) and set a_i^X to be its only element, for $i = 1, \dots, n$. It is easy to see that (73) holds.

Now suppose $\boldsymbol{\iota}_1, \dots, \boldsymbol{\iota}_k$ are the equations in X and $\neg\boldsymbol{\iota}'_1, \dots, \neg\boldsymbol{\iota}'_m$ are the negated equations in X, for $m \geq 1$ (we can always assume that $k \geq 1$). For each j, $1 \leq j \leq m$, take the sp-type quasiequation

$$\boldsymbol{\rho}_j \ = \ (\boldsymbol{\iota}_1 \,\&\, \dots \,\&\, \boldsymbol{\iota}_k \Rightarrow \boldsymbol{\iota}'_j)(p_1/a_1, \dots, p_n/a_n).$$

Then $\mathfrak{A} \not\models \boldsymbol{\rho}_j$, and so, since rL is BAO-rule-conservative, there is some $\mathfrak{B}_j \in \mathsf{BAO}$ with $\mathfrak{B}_j \models rL$ and $\mathfrak{B}_j \not\models \boldsymbol{\rho}_j$. Then there are $b_1^j, \dots, b_n^j$ in $\mathfrak{B}_j$ such that

$$\mathfrak{B}_j \models \Big(\bigwedge_{i=1}^{k} \boldsymbol{\iota}_i \wedge \neg\boldsymbol{\iota}'_j\Big)(p_1/a_1, \dots, p_n/a_n)[b_1^j, \dots, b_n^j].$$

Now let $\mathfrak{B}^X = \prod_{j=1}^m \mathfrak{B}_j$ and $a_i^X = (b_i^1, \dots, b_i^m)$, for $i = 1, \dots, n$. Then clearly we have (73). As the class BAO_{rL} is a quasivariety, it is closed under direct products, and so $\mathfrak{B}^X \in \mathsf{BAO}_{rL}$ as required. ⊣

Let $T_\mathfrak{A}$ be the set of all finite subsets of $D_\mathfrak{A}$. For every $X \in T_\mathfrak{A}$, let

$$\mathcal{J}_X = \{Y \in T_\mathfrak{A} \mid X \subseteq Y\}.$$

As $X_1 \cup \dots \cup X_m \in \mathcal{J}_{X_1} \cap \dots \cap \mathcal{J}_{X_m}$, the collection $\{\mathcal{J}_x \mid X \in T_\mathfrak{A}\}$ has the finite intersection property, and so there is an ultrafilter U over $T_\mathfrak{A}$ extending

$\{\mathcal{J}_x \mid X \in T_{\mathfrak{A}}\}$. For $X \in T_{\mathfrak{A}}$, take the BAO $\mathfrak{B}^X$ given by Claim 63.1, and let

$$\mathfrak{B} = \prod_{X \in T_{\mathfrak{A}}} \mathfrak{B}^X / U.$$

As the class BAO_{rL} is a quasivariety, it is closed under ultraproducts, and so $\mathfrak{B} \in \mathsf{BAO}_{rL}$. Define an $\eta\colon \mathfrak{A} \to \mathfrak{B}$ map by taking $\eta(a) = [(\hat{a}^X)_{X \in T_{\mathfrak{A}}}]_U$, where for all a in $\mathfrak{A}$ and $X \in T_{\mathfrak{A}}$,

$$\hat{a}^X = \begin{cases} a^X, & \text{if } a \text{ occurs in some literal in } X, \\ \text{arbitrary element of } \mathfrak{B}^X, & \text{otherwise.} \end{cases}$$

By Claim 63.1 and Łos' Theorem [23], for every $\varphi(a_1, \dots, a_n) \in D_{\mathfrak{A}}$, we have

$$\mathfrak{B} \models \varphi(p_1/a_1, \dots, p_n/a_n)[\eta(a_1), \dots, \eta(a_n)].$$

Thus, η is an sp-embedding from $\mathfrak{A}$ into the sp-type reduct of $\mathfrak{B}$. $\dashv$

We call an spi-rule logic *rL* BAO-*complex* if the sp-type reduct of every $\mathfrak{A} \in \mathsf{BAO}_{rL}$ is embeddable into some $\mathfrak{F}^\star$ with $\mathfrak{F} \in \mathsf{Kr}_{rL}$. Note that, as sp-implications correspond to Sahlqvist formulas in modal logic, any spi-logic L is BAO-complex. As a consequence of Lemma 63 we obtain:

THEOREM 64. *For every* BAO-*complex spi-rule logic rL, the following are equivalent:*

(*i*) *rL is complex;*
(*ii*) *rL is globally complete;*
(*iii*) *rL is* BAO-*rule-conservative;*
(*iv*) *rL is* BAO-*embeddable.*

PROOF. (i) $\Rightarrow$ (ii) follows from (72); (ii) $\Rightarrow$ (iii) is trivial; (iii) $\Rightarrow$ (iv) follows from Lemma 63; and (iv) $\Rightarrow$ (i) follows from the fact that *rL* is BAO-complex. $\dashv$

§10. Conclusion. In this article, we have started developing the completeness theory of spi-logics. Of course, many interesting and challenging problems remain to be explored. A few concrete open questions have already been mentioned above, and there is a more or less standard list of problems regarding properties of modal logics and their lattices; see, e.g., [22, 13, 77, 14]. Here, we briefly discuss few possible directions of follow-up research.

(1) In Boolean modal logic, the *degree of Kripke incompleteness* of a normal modal logic Λ—that is, the cardinality of the set of normal modal logics whose Kripke frames coincide with the Kripke frames of Λ [32]—has been used to analyse the position of Kripke incomplete logics within the lattice of all normal modal logics. Wim Blok [15] established the following dichotomy: the degree of Kripke incompleteness of a consistent normal unimodal logic Λ is either $2^{\aleph_0}$ or 1, in which case Λ is a union of co-splitting logics; see also [55, 77, 53]. Given this complete classification, the question arises as to whether one can also characterise the degree of Kripke incompleteness of spi-logics and whether this is again linked to co-splittings (now in the lattice of spi-logics) and the existence of some analogue of Jankov-Fine formulas [47, 31].

(2) To prove undefinability of frame classes by sp-implications, we developed a necessary condition for frame definability. In Boolean modal logic, the

Goldblatt–Thomason theorem [41] provides necessary and sufficient conditions for frame definability in terms of p-morphisms, generated subframes, disjoint unions, and ultrafilter extensions. Can one give natural necessary and sufficient conditions for frame definability by sp-implications?

(3) It is readily seen that spi-rules can define non-elementrary frame conditions and thus behave differently from sp-implications [54]. We have also seen that complex spi-rule logics are exactly those that are globally complete. Thus, it would be interesting to extend the completeness theory of spi-logics developed in this paper to spi-rule logics.

(4) The embeddability of SLOs into full complex algebras of Kripke frames is shown by Sofronie-Stokkermans [70, 71] using a method that is different from those in §§4.1.1–4.1.2 and involves *distributive lattices with normal and* $\vee$*-additive operators* (DLOs). A given SLO $\mathfrak{A}$ is first embedded into the DLO $\mathfrak{A}^{\vee}$ of its downsets, which is then embedded into the full complex algebra of some frame $\mathfrak{F}$ over the prime filters of $\mathfrak{A}^{\vee}$ using Goldblatt's [40] extension of Priestley duality [61] to operators. She also shows that validity of sp-implications of the form $\Diamond_1 \dots \Diamond_n p \to \Diamond_0 p$ transfers from $\mathfrak{A}$ to $\mathfrak{F}$. It would be interesting to study the boundaries of this method and its connections to §§4.1.1–4.1.2. More generally, one can ask which sp-implications are SLO–to–DLO- and/or DLO–to–BAO-conservative? The latter question can also be investigated for spi$^{\vee}$-implications, that is, implications between sp-formulas with disjunction.

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